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Weak thresholding greedy algorithms in Banach spaces ☆

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Abstract

We consider weak thresholding greedy algorithms with respect to Markushevich bases in general Banach spaces. We find sufficient conditions for the equivalence of boundedness and convergence of the approximants. We also show that if there is a weak thresholding algorithm for the system which gives the best n-term approximation up to a multiplicative constant, then the system is already "greedy". Similar results are proved for "almost greedy" and "semi-greedy" systems.

Keywords: Greedy approximation; Banach spaces; Thresholding greedy algorithm; Weak thresholding

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Contents

1. Introduction

Let X be an infinite-dimensional Banach space and let (e_i) be a Markushevich basis for X with biorthogonal sequence (e_i^*). The *Thresholding Greedy Algorithm* (TGA) was introduced by Temlyakov [11] for the trigonometric system and subsequently extended to the Banach space setting by Konyagin and Temlyakov [8]. See [13] and the recent monograph [14] for the history of the problem and for background information on greedy approximation. The algorithm is defined as follows. For $x \in X$ and $n \geqslant 1$, let $\mathcal{A}_n(x) \subset \mathbb{N}$ be the indices corresponding to a choice of n largest coefficients of x in absolute value, i.e. $\mathcal{A}_n(x)$ satisfies

$$\min\{|e_i^*(x)|: i \in \mathcal{A}_n(x)\} \geqslant \max\{|e_i^*(x)|: i \in \mathbb{N} \setminus \mathcal{A}_n(x)\}.$$

Then $\mathcal{G}_n(x) := \sum_{i\in\mathcal{A}_n(x)} e_i^*(x)e_i$ is called an nth *greedy approximant* to x. The TGA is said to converge if $\mathcal{G}_n(x) \to x$. We say that (e_i) is *quasi-greedy* (QG) if there exists $K < \infty$ such that for all $x \in X$ and $n \geqslant 1$, we have $\|\mathcal{G}_n(x)\| \leqslant K\|x\|$. Wojtaszczyk [15, Theorem 1] proved that (e_i) is QG if and only if the TGA converges for all initial vectors $x \in X$.

It is known [4, Remark 6.3] that the Haar basis (normalized in $L_1[0,1]$) is not quasi-greedy, i.e., that for certain initial vectors x the TGA does not converge. Recently, however, Gogyan discovered a *weak thresholding* version of the TGA for the Haar basis which converges. The new algorithm defined in [7] is of the following general type which we call *branch greedy*. Fix a *weakness parameter* τ with $0 < \tau < 1$. For each $x \in X$, we define inductively an increasing sequence $(\mathcal{A}_n^\tau(x))$ of sets of n coefficient indices such that

$$\min\{|e_i^*(x)|: i \in \mathcal{A}_n^\tau(x)\} \geqslant \tau \max\{|e_i^*(x)|: i \in \mathbb{N} \setminus \mathcal{A}_n^\tau(x)\}.$$

Such index sets will be generated by a weak thresholding procedure. Gogyan proved for his algorithm that the *branch greedy approximants* $\mathcal{G}_n^\tau(x) := \sum_{i\in\mathcal{A}_n^\tau(x)} e_i^*(x)e_i$ converge to x and are uniformly bounded, i.e., that $\|\mathcal{G}_n^\tau(x)\| \leqslant K(\tau)\|x\|$, where $K(\tau)$ is a constant.

The motivation for the term "branch greedy" comes from the fact that weak thresholding generates a tree of possible choices for the coefficients. A branch greedy algorithm is simply a procedure for selecting a branch of this tree. In Section 2 we try to formulate a reasonable and widely applicable definition of branch greedy algorithm. Then our goal is to study convergence and best approximation properties of branch greedy algorithms. In Section 3 we study the analogue of Wojtaszczyck's theorem on the equivalence of convergence of the TGA and the

QG property [15]. We identify some sufficient conditions for this equivalence to hold for branch greedy algorithms and we show that the equivalence does not hold in general.

In Section 4 we define a system to be *branch quasi-greedy* (BQG) if the branch greedy approximants are uniformly bounded. Gogyan's result shows that the Haar basis for $L_1[0,1]$ is BQG but not QG. We show that the QG property is equivalent to the BQG property together with an additional "partial unconditionality" type condition (see e.g. [5]). This fact is used repeatedly in the subsequent sections. As an application of this result we show that every weakly null BQG sequence contains a QG subsequence, which sheds some light on an important open problem concerning partial unconditionality.

The remainder of the paper concerns the best n-term approximation for branch greedy algorithms. Recall that the error in the best n-term approximation to x (using (e_i)) is given by

$$\sigma_n(x) := \inf\left\{\left\|x - \sum_{i\in A} a_i e_i\right\|:\ (a_i)\subset\mathbb{R},\ |A| = n\right\},$$

and the error in the best projection of x onto a subset of (e_i) of size at most n is given by

$$\tilde{\sigma}_n(x) := \inf\left\{\left\|x - \sum_{i\in A} e_i^*(x) e_i\right\|:\ |A| \leqslant n\right\}.$$

Then (e_i) is said to be *greedy* with constant C [8] if

$$\|x - \mathcal{G}_n(x)\| \leqslant C\sigma_n(x) \quad (n \geqslant 1,\ x\in X),$$

and *almost greedy* (AG) with constant C [3] if

$$\|x - \mathcal{G}_n(x)\| \leqslant C\tilde{\sigma}_n(x) \quad (n \geqslant 1,\ x\in X).$$

Temlyakov [12] proved that the Haar system for $L_p[0,1]^d$ ($1 < p < \infty$, $d \geqslant 1$) is greedy, which provides an important theoretical justification for the use of thresholding in data compression. We refer the reader to [16] for other examples of greedy bases.

Konyagin and Temlyakov [8] gave a very useful characterization of greedy bases. They proved that a system is greedy if and only if it is *unconditional* and *democratic*. The democratic property is defined as follows. We say that (e_i) is *democratic* with constant Δ if, for all finite $A, B \subset \mathbb{N}$ with $|A| \leqslant |B|$, we have

$$\left\|\sum_{i\in A} e_i\right\| \leqslant \Delta\left\|\sum_{i\in B} x_i\right\|.$$

We recall that (e_i) is *unconditional* with constant K if, for all choices of signs, we have

$$\left\|\sum_{i=1}^{\infty} \pm e_i^*(x) e_i\right\| \leqslant K\|x\| \quad (x\in X).$$

We introduce the classes of *branch greedy* systems in Section 5 and *branch almost greedy* systems in Section 6. It turns out, however, that the characterizations discussed above remain valid, so that the class of branch greedy (resp. branch almost greedy) systems coincides with the class of greedy (resp. almost greedy) systems. At the expense of some extra complexity in the proofs, we have formulated all our results for *finite* systems, thereby avoiding infinite-dimensional arguments that are not valid in the finite-dimensional setting. In particular, we obtain quantitative estimates that are independent of dimension for many of the various constants that arise: for example, we can estimate the democratic and quasi-greedy constants in terms of the branch almost greedy constant and the weakness parameter τ. None of our estimates here involve the *basis constant* of the system. It follows that our infinite-dimensional results are valid for general biorthogonal systems.

The last section, which is the most technical, concerns the branch analogue of the notion of *semi-greedy* system introduced in [2]. Let us recall that (e_i) is semi-greedy with constant C if for all $x \in X$ and $n \geqslant 1$ there exist scalars a_i $(i \in \mathcal{A}_n(x))$ such that

$$\left\| x - \sum_{i \in A_n(x)} a_i e_i \right\| \leqslant C\sigma_n(x).$$

Several questions remain open for semi-greedy systems. In particular, we are not able to show without extra hypotheses that a branch semi-greedy system is semi-greedy. Moreover, our quantitative results involve the basis constant of the system and in some cases also the *cotype q constant* of X. In the infinite-dimensional setting we can show that if X has finite cotype and (e_i) is a branch semi-greedy Schauder basis then (e_i) is almost greedy. This implies the equivalence of the semi-greedy and branch semi-greedy properties for Schauder bases of spaces with finite cotype.

2. Branch greedy algorithms

Let X be a finite-dimensional or separable infinite-dimensional Banach space. Let $(e_i) \subset X$ be a semi-normalized system, that is, $a \leqslant \|e_i\| \leqslant b$ for positive constants a and b. We assume that (e_i) is a *bounded Markushevich basis* for X with biorthogonal functionals (e_i^*), that is, $\sup_{i \geqslant 1} \|e_i^*\| = M < \infty$, (e_i) has dense linear span, and (e_i^*) is total, i.e. the mapping sending $x \in X$ to its coefficient sequence $(e_i^*(x))$ is one–one. The support of x is defined by $\operatorname{supp}(x) := \{i\colon e_i^*(x) \neq 0\}$. For every finite $A \subset \mathbb{N}$, we denote by P_A the projection $P_A(x) := \sum_{i \in A} e_i^*(x)e_i$. If A is co-finite, we define $P_A := I - P_{\mathbb{N}\setminus A}$. Now fix a weakness parameter τ with $0 < \tau < 1$. For all $0 \neq x \in X$, define

$$\mathcal{A}^\tau(x) := \left\{ i \in \mathbb{N}\colon \left|e_i^*(x)\right| \geqslant \tau \max_{i \geqslant 1}\left|e_i^*(x)\right| \right\}.$$

Let $\mathcal{G}^\tau\colon X \setminus \{0\} \to \mathbb{N}$ be any mapping which satisfies the following conditions:

(a) $\mathcal{G}^\tau(x) \in \mathcal{A}^\tau(x)$;
(b) $\mathcal{G}^\tau(\lambda x) = \mathcal{G}^\tau(x)$ for all $\lambda \neq 0$;
(c) If $\mathcal{A}^\tau(y) = \mathcal{A}^\tau(x)$ and $e_i^*(y) = e_i^*(x)$ for all $i \in \mathcal{A}^\tau(x)$ then $\mathcal{G}^\tau(y) = \mathcal{G}^\tau(x)$.

Here $\mathcal{G}^\tau(x)$ is to be interpreted as the index of the first coefficient of x that is selected by the algorithm. Subsequent coefficients are then selected by iterating the algorithm on the residuals. Condition (a) simply says that the coefficients are selected by weak thresholding with weakness parameter τ. Condition (b) is a natural homogeneity assumption. Condition (c) says that the choice of the next coefficient should depend only on the set of coefficients (indexed by $\mathcal{A}^\tau(x)$) which satisfy the weak thresholding criterion.

Every such mapping $\mathcal{G}^\tau$ generates a "branch greedy algorithm" as follows. For each finitely supported (resp., infinitely supported) vector $x \in X$ define the "branch greedy ordering" ρ_x^τ: $\{1, 2, \ldots, |\operatorname{supp}(x)|\} \to \mathbb{N}$ (resp., $\rho_x^\tau : \mathbb{N} \to \mathbb{N}$) inductively:

(i) If $x \neq 0$ then $\rho_x^\tau(1) = \mathcal{G}^\tau(x)$;
(ii) For $i \geqslant 2$, if $|\operatorname{supp}(x)| \geqslant 2$ then

$$\rho_x^\tau(i) = \mathcal{G}^\tau\left(x - \sum_{j=1}^{i-1} e^*_{\rho_x^\tau(j)}(x) e_{\rho_x^\tau(j)}\right).$$

Henceforth we shall drop the subscript x from ρ_x^τ when there is no ambiguity. Finally, we define the *branch greedy approximations* $\mathcal{G}_n^\tau(x)$ as follows (setting $e^*_{\rho^\tau(i)}(x) := 0$ if $i > |\operatorname{supp}(x)|$):

$$\mathcal{G}_n^\tau(x) = \sum_{i=1}^n e^*_{\rho^\tau(i)}(x) e_{\rho^\tau(i)} \quad (n \geqslant 1).$$

Set $\mathcal{G}_0^\tau(x) := 0$ for convenience.

Note that $(\rho^\tau(i))$ is a generalization of the *greedy ordering* $(\rho(i))$, which corresponds to $\tau = 1$ and simply rearranges the coefficients of x in decreasing order of magnitude:

$$\left|e^*_{\rho(1)}(x)\right| \geqslant \left|e^*_{\rho(2)}(x)\right| \geqslant \cdots,$$

choosing the smallest index in case of a tie [2, p. 577].

3. Convergence

In this section we consider the following two desirable properties of the branch greedy algorithms defined above:

(A) Convergence of the algorithm, i.e., $\mathcal{G}_n^\tau(x) \to x$ for all $x \in X$.
(B) Uniform boundedness of the approximants, i.e., there exists $K < \infty$ such that $\|\mathcal{G}_n^\tau(x)\| \leqslant K\|x\|$ for all $n \geqslant 1$ and $x \in X$.

Proposition 3.1. (A) *and* (B) *are not equivalent in general.*

Proof. It suffices to observe that Gogyan's algorithm for the normalized Haar basis in $L_1[0,1]$ (see [7] or Example 3.2 below) can be modified slightly so that the modified algorithm satisfies (A) but not (B). Rather than giving a precise definition of the mapping $\mathcal{G}^\tau$ we give a more informal description of the algorithm. To that end, let (f_k) be the leftmost branch of the Haar basis, i.e., $f_k = 2^{k-1}(\chi_{[0,2^{-k}]} - \chi_{(2^{-k},2^{1-k}]})$. First we modify the algorithm for vectors of the form $x_n = \sum_{k=1}^{2n} f_k$ $(n \geqslant 1)$. For such special vectors and their scalar multiples we modify the definition of the mapping $\mathcal{G}^\tau$ so that the first n branch greedy approximants of x_n are given by $\mathcal{G}_k^\tau(x_n) := \sum_{j=1}^k f_{2j-1}$ for $1 \leqslant k \leqslant n$. To ensure that conditions (b) and (c) remain satisfied it is also necessary to modify the definition of $\mathcal{G}^\tau$ accordingly on all scalar multiples of vectors of the form $x_n - \mathcal{G}_k^\tau(x_n) + y$ $(1 \leqslant k \leqslant n-1)$, where $x_n - \mathcal{G}_k^\tau(x_n)$ and y are disjointly supported with respect to the Haar basis and all the Haar coefficients of y are smaller than τ. For all other vectors the definition of $\mathcal{G}^\tau$ is unchanged. The modified algorithm satisfies conditions (a)–(c) of Section 2, but $\|x_n\| \leqslant 2$ and $\|\mathcal{G}_n^\tau(x_n)\| \geqslant n/4$ (see [4, Remark 6.3]). Hence property (B) does not hold. However, for each fixed $x \in L_1[0,1]$, there is at most one value of n for which a residual of x for Gogyan's algorithm will be equal to a scalar multiple of a vector of the form $x_n - \mathcal{G}_k^\tau(x_n) + y$. So after finitely many iterations the modified algorithm coincides with Gogyan's algorithm and hence converges to x. So property (A) holds. $\square$

The main result of this section is that (A) and (B) are equivalent for a natural class of branch greedy algorithms. To that end we consider two conditions:

(H1) For all finite $A \subset \mathbb{N}$ there exists a finite $\overline{A} \subset \mathbb{N}$ such that $A \subseteq \overline{A}$ and for all $x, y \in X$ such that $\operatorname{supp}(x-y) \subset A$, we have that for all $m \geqslant 1$ there exists $n \geqslant 1$ such that

$$\mathcal{G}_m^\tau(x)|_{\mathbb{N}\setminus\overline{A}} = \mathcal{G}_n^\tau(y)|_{\mathbb{N}\setminus\overline{A}},$$

i.e. $\mathcal{G}_m^\tau(x)$ and $\mathcal{G}_n^\tau(y)$ agree on the complement of $\overline{A}$.

(H2) For all $x, y \in X$, if $\operatorname{supp}(x-y)$ is finite then there exist $m_1, m_2 \in \mathbb{N}$ such that $x - \mathcal{G}_{m_1}^\tau(x) = y - \mathcal{G}_{m_2}^\tau(y)$. (Note that this implies that for all $k \geqslant 0$, $x - \mathcal{G}_{m_1+k}^\tau(x) = y - \mathcal{G}_{m_2+k}^\tau(y)$.)

It is easily seen that the TGA (using the greedy ordering) satisfies (H1), with $\overline{A} = A$, and (H2). The branch greedy algorithm for the Haar system in $L_1[0,1]$ defined by Gogyan [7] belongs to the following class of algorithms satisfying (H1).

Example 3.2. Let $\prec$ be any tree ordering on $\mathbb{N}$, i.e. $\prec$ is a partial order such that for every $n \in \mathbb{N}$ the initial segment $\{m \in \mathbb{N}\colon m \prec n\}$ is finite and totally ordered. If $m \preccurlyeq n$, let $[m,n]$ denote the segment $\{i \in \mathbb{N}\colon m \preccurlyeq i \preccurlyeq n\}$. For $x \in X$, recall that $\rho_x(1)$ is the smallest integer i_0 at which $i \mapsto |e_i^*(x)|$ is maximized. We define $\mathcal{G}^\tau(x)$ as follows: $\mathcal{G}^\tau(x) \preccurlyeq \rho_x(1)$ and $[\mathcal{G}^\tau(x), \rho_x(1)]$ is the largest segment such that $|e_i^*(x)| \geqslant \tau |e_{\rho_x(1)}^*(x)|$ for all $i \in [\mathcal{G}^\tau(x), \rho_x(1)]$.

Proposition 3.3. *The branch greedy algorithm determined by $\mathcal{G}^\tau$ in Example 3.2 satisfies* (H1).

Proof. For a given finite set $A \subset \mathbb{N}$, let $\overline{A} := \bigcup_{i \in A}\{j \in \mathbb{N}\colon j \preccurlyeq i\}$. Clearly, $\overline{A}$ is finite. Suppose that $x, y \in X$ satisfy $\operatorname{supp}(x-y) \subseteq A$. We shall prove by induction that for all $m \geqslant 0$ there exists $n_m \geqslant 1$ such that

$$\mathcal{G}_m^\tau(x)|_{\mathbb{N}\setminus\overline{A}} = \mathcal{G}_{n_m}^\tau(y)|_{\mathbb{N}\setminus\overline{A}}. \tag{1}$$

Setting $n_0 := 0$ handles the case $m = 0$. Suppose the result holds for m. If $\rho_x^\tau(m+1) \in \overline{A}$ then $\mathcal{G}_{m+1}^\tau(x)|_{\mathbb{N}\setminus\overline{A}} = \mathcal{G}_m^\tau(x)|_{\mathbb{N}\setminus\overline{A}}$, so we can take $n_{m+1} := n_m$. If $\rho_x^\tau(m+1) \in \mathbb{N} \setminus \overline{A}$, let

$$n_{m+1} := \min\bigl\{k > n_m\colon \rho_y^\tau(k) \in \mathbb{N} \setminus \overline{A}\bigr\}.$$

It follows easily from the definition of $\mathcal{G}^\tau$ and (1) that $\rho_x^\tau(m+1) = \rho_y^\tau(n_{m+1})$, which gives (1) with m replaced by $m+1$. $\square$

Remark 3.4. It is an instructive exercise to check that the algorithm described in Proposition 3.1 does not satisfy (H1) and hence is not in contradiction with Theorem 3.7 below.

Lemma 3.5. (H1) $\Rightarrow$ (H2).

Proof. Suppose that $x, y \in X$ and that $A := \operatorname{supp}(x - y)$ is finite. We may assume that $\operatorname{supp}(x)$ is infinite, otherwise (H2) is trivially satisfied. Let $\overline{A}$ be the set postulated by (H1). Let

$$\delta := \min\{|e_i^*(x)|, |e_i^*(y)|: i \in \overline{A} \cap \operatorname{supp}(x),\ j \in \overline{A} \cap \operatorname{supp}(y)\}. \tag{2}$$

Since $\operatorname{supp}(x)$ is infinite, we may choose $m \in \mathbb{N}$ such that $0 < |e^*_{\rho^\tau(m)}(x)| < \tau\delta/2$. Clearly, $i_0 := \rho^\tau(m) \notin \overline{A}$. By (H1) there exists $k \in \mathbb{N}$ such that

$$\mathcal{G}_k^\tau(y)|_{\mathbb{N}\setminus\overline{A}} = \mathcal{G}_m^\tau(x)|_{\mathbb{N}\setminus\overline{A}}. \tag{3}$$

Since $i_0 \notin \overline{A}$, we have

$$0 < |e_{i_0}^*(\mathcal{G}_k^\tau(y))| = |e_{i_0}^*(\mathcal{G}_m^\tau(x))| < \frac{\tau\delta}{2},$$

which when combined with (2) implies that

$$\mathcal{G}_m^\tau(x)|_{\overline{A}} = x|_{\overline{A}} \quad \text{and} \quad \mathcal{G}_k^\tau(y)|_{\overline{A}} = y|_{\overline{A}}.$$

Combining the latter with (3) and using the fact that $\operatorname{supp}(x - y) = A \subset \overline{A}$, we deduce that $x - \mathcal{G}_m^\tau(x) = y - \mathcal{G}_k^\tau(y)$. Hence (H2) is satisfied. $\square$

Proposition 3.6. *If the algorithm satisfies* (H2) *then* (B) $\Rightarrow$ (A).

Proof. Suppose that (H2) and (B) hold. Given $x \in X$ and $\varepsilon > 0$, choose a finitely supported $z \in X$ with $\|x - z\| < \varepsilon$. Applying (H2) to x and to $y := x - z$ (noting that $\operatorname{supp}(y - x) = \operatorname{supp}(z)$ is finite) there exist $m_1, m_2 \in \mathbb{N}$ such that for all $k \geqslant 0$

$$x - \mathcal{G}_{m_1+k}^\tau(x) = x - z - \mathcal{G}_{m_2+k}^\tau(x - z).$$

Now applying (B), we have that for all $k \geqslant 0$,

$$\|x - \mathcal{G}_{m_1+k}^\tau(x)\| \leqslant \|x - z\| + \|\mathcal{G}_{m_2+k}^\tau(x - z)\| \leqslant (1 + K)\varepsilon.$$

Hence $\mathcal{G}_n^\tau(x) \to x$. $\square$

The following theorem generalizes [15, Theorem 1].

Theorem 3.7. *If the algorithm satisfies* (H1) *then* (A) $\Leftrightarrow$ (B).

Proof. Suppose that (H1) is satisfied. Then (B) $\Rightarrow$ (A) follows from Lemma 3.5 and Proposition 3.6.

For the converse, we shall assume that (A) holds but that (B) does not hold and obtain a contradiction. First we claim that given a finite $A \subset \mathbb{N}$ and given $K > 0$ there exist a finite set $B \subset \mathbb{N}$ disjoint from A and $x \in X$ such that $\|x\| = 1$, $\operatorname{supp}(x) \subset B$, and $\|\mathcal{G}_k^\tau(x)\| \geqslant K$ for some $k \geqslant 1$. Let $\overline{A}$ be the finite set given by (H1) and let M be the maximum of the norms of the (finitely many) finite-dimensional projections P_Ω ($\Omega \subset \overline{A}$). Since (B) does not hold, given $K_1 > 0$ there exists $x_1 \in X$ such that $\|x_1\| = 1$ and $\|\mathcal{G}_m^\tau(x_1)\| \geqslant K_1$ for some $m \geqslant 1$. Let $x_2 = x_1 - P_A(x_1)$. Then $\|x_2\| \leqslant \|x_1\| + \|P_A\|\|x_1\| \leqslant 1 + M$. Since $\operatorname{supp}(x_1 - x_2) \subset A$, it follows from (H1) that there exists $k \in \mathbb{N}$ such that

$$P_{\mathbb{N}\setminus\overline{A}}\big(\mathcal{G}_k^\tau(x_2)\big) = P_{\mathbb{N}\setminus\overline{A}}\big(\mathcal{G}_m^\tau(x_1)\big).$$

Now $P_{\overline{A}}(\mathcal{G}_k^\tau(x_2)) = P_\Omega(x_1)$, where $\Omega = \overline{A} \cap \operatorname{supp}(\mathcal{G}_k^\tau(x_2))$, so

$$\big\|P_{\overline{A}}\big(\mathcal{G}_k^\tau(x_2)\big)\big\| \leqslant M\|x_1\| = M.$$

Hence

$$\begin{aligned}
\big\|\mathcal{G}_k^\tau(x_2)\big\| &\geqslant \big\|P_{\mathbb{N}\setminus\overline{A}}\big(\mathcal{G}_k^\tau(x_2)\big)\big\| - \big\|P_{\overline{A}}\big(\mathcal{G}_k^\tau(x_2)\big)\big\| \\
&= \big\|P_{\mathbb{N}\setminus\overline{A}}\big(\mathcal{G}_m^\tau(x_1)\big)\big\| - \big\|P_{\overline{A}}\big(\mathcal{G}_k^\tau(x_2)\big)\big\| \\
&\geqslant \big\|P_{\mathbb{N}\setminus\overline{A}}\big(\mathcal{G}_m^\tau(x_1)\big)\big\| - M \\
&\geqslant \big\|\mathcal{G}_m^\tau(x_1)\big\| - 2M \\
&\geqslant K_1 - 2M.
\end{aligned}$$

Let $x_3 = x_2/\|x_2\|$. Then $\|\mathcal{G}_k^\tau(x_3)\| \geqslant (K_1 - 2M)/(M + 1)$. Let

$$\delta := \min\big\{\big|e_i^*\big(\mathcal{G}_k^\tau(x_3)\big)\big|:\ i \in \operatorname{supp}\big(\mathcal{G}_k^\tau(x_3)\big)\big\}. \tag{4}$$

Choose a finite set $B_1 \subset \mathbb{N}$ such that

$$\big|e_i^*(x_3)\big| \leqslant \frac{\tau\delta}{2} \quad (i \in \mathbb{N} \setminus B_1). \tag{5}$$

Note that $\operatorname{supp}(\mathcal{G}_k^\tau(x_3)) \subseteq B_1$. Since $\operatorname{supp}(x_3)$ is disjoint from A and since $A \cup B_1$ is finite, it follows that, given $\eta > 0$, using the fact that (e_i) is a bounded Markushevich basis, we may choose a finite set B disjoint from A with $B_1 \subseteq B$, and we may choose $x_4 \in X$ such that $\operatorname{supp}(x_4) \subseteq B$, $\|x_3 - x_4\| < \eta$, and

$$x_4|_{B_1} = x_3|_{B_1}. \tag{6}$$

It follows that for all $i \in \mathbb{N} \setminus B_1$, we have

$$\big|e_i^*(x_4)\big| \leqslant \big|e_i^*(x_3)\big| + \Big(\sup_{i\in\mathbb{N}}\big\|e_i^*\big\|\Big)\|x_3 - x_4\| \leqslant \frac{\tau\delta}{2} + \Big(\sup_{i\in\mathbb{N}}\big\|e_i^*\big\|\Big)\eta \leqslant \frac{3\tau\delta}{4} \tag{7}$$

provided η is sufficiently small. It follows from (4), (5), (6), and (7) that for $0 \leqslant j < k$,

$$\mathcal{A}^\tau\big(x_3 - \mathcal{G}_j^\tau(x_3)\big) = \mathcal{A}^\tau\big(x_4 - \mathcal{G}_j^\tau(x_4)\big).$$

This implies, in conjunction with (6) and with assumption (c) concerning ρ^τ, that

$$\mathcal{G}_j^\tau(x_4) = \mathcal{G}_j^\tau(x_3) \quad (1 \leqslant j \leqslant k).$$

Hence $\|\mathcal{G}_k^\tau(x_4)\| \geqslant (K_1 - 2M)/(M+1)$ and $\|x_4\| \leqslant 1 + \eta$. Since K_1 can be chosen arbitrarily large, $x_5 = x_4/\|x_4\|$ verifies the claim.

Having established the claim we can choose disjointly and finitely supported vectors x_n and positive integers k_n $(n \geqslant 1)$ such that $\|x_n\| \leqslant 2^{-n}$, $\|\mathcal{G}_{k_n}(x_n)\| \geqslant n$, and

$$\max\big\{\big|e_i^*(x_{n+1})\big|: i \in \mathbb{N}\big\} \leqslant \frac{\tau}{2}\min\big\{\big|e_i^*(x_n)\big|: i \in \operatorname{supp}(x_n)\big\}.$$

Let $x = \sum_{i=1}^\infty x_i$ and let $m_i = |\operatorname{supp}(x_i)|$. Clearly, for $n \geqslant 1$,

$$\mathcal{G}^\tau_{m_1+\cdots+m_n+k_{n+1}}(x) = x_1 + \cdots + x_n + \mathcal{G}^\tau_{k_{n+1}}(x_{n+1}).$$

Hence

$$\begin{aligned}\big\|\mathcal{G}^\tau_{m_1+\cdots+m_n+k_{n+1}}(x)\big\| &\geqslant \big\|\mathcal{G}^\tau_{k_{n+1}}(x_{n+1})\big\| - \sum_{i=1}^\infty \|x_i\| \\ &\geqslant (n+1) - 1 = n.\end{aligned}$$

This is the desired contradiction to (A). $\square$

4. Branch quasi-greedy systems

Henceforth we shall formulate most of our results in the finite-dimensional setting for greater precision. Let $(e_i)_{i=1}^N$ be an algebraic basis for an N-dimensional normed space X. Let $(e_i^*)_{i=1}^N$ be the corresponding biorthogonal functionals. We shall assume as above that $a \leqslant \|e_i\| \leqslant b$ for fixed positive constants a and b. For a fixed weakness parameter $\tau \in (0,1)$, we shall consider a branch greedy algorithm determined by a mapping $\mathcal{G}^\tau$ as described above.

Definition 4.1. We say that (e_i) is *branch quasi-greedy* with weakness parameter τ $(BQG(\tau))$ and constant K if for all $x \in X$ and $0 \leqslant k \leqslant N$, we have

$$\big\|\mathcal{G}_k^\tau(x)\big\| \leqslant K\|x\|.$$

We begin with an important observation which shows that the definition of a $BQG(\tau)$ system is only meaningful for bounded biorthogonal systems.

Proposition 4.2. *Suppose that $(e_i)_{i=1}^N$ is $BQG(\tau)$ with constant K. Then $\|e_i^*\| \leqslant \frac{K}{a\tau}$ for all i.*

Proof.

$$\left|e_i^*(x)\right| \leqslant \frac{1}{\tau}\left|e_{\rho^\tau(1)}^*(x)\right| \leqslant \frac{\|\mathcal{G}_1^\tau(x)\|}{\tau a} \leqslant \frac{K\|x\|}{\tau a}. \qquad \square$$

There exist BQG systems that are not QG, e.g. the Haar basis for $L_1[0,1]$ [7]. The following slightly technical definition captures the gap between the two notions.

Definition 4.3. Let $0<\tau<1$. Then $(e_j)_{j=1}^N$ has property $P(\tau)$ if there exists $C<\infty$ such that for all $A\subset\{1,\dots,N\}$ and for all scalars $(a_i)_{i\in A}$, with $1\leqslant|a_i|\leqslant 1/\tau^2$, we have

$$\max_{\pm}\left\|\sum_{i\in A}\pm e_i\right\| \leqslant C\left\|\sum_{i\in A}a_ie_i\right\|. \tag{8}$$

Proposition 4.4. *Suppose that $(e_i)_{1\leqslant i\leqslant N}$ is $BQG(\tau)$ with constant K and has property $P(\tau)$ with constant C. Then (e_i) is QG with constant $K(1+\frac{2C}{\tau^2})$.*

Proof. Fix $n\geqslant 1$, and let m be the least integer such that $(\rho(i))_{i=1}^n \subseteq (\rho^\tau(i))_{i=1}^m$. Thus, either $\mathcal{G}_m^\tau(x)=\mathcal{G}_n(x)$ (in which case $\|\mathcal{G}_n(x)\|\leqslant K\|x\|$) or

$$\mathcal{G}_m^\tau(x)=\mathcal{G}_n(x)+\sum_{i=k}^{l}\eta_i a_{\rho^\tau(i)}e_{\rho^\tau(i)},$$

where

$$k=\min\bigl\{i\leqslant m:\ \rho^\tau(i)\notin\bigl\{\rho(1),\rho(2),\dots,\rho(n)\bigr\}\bigr\},$$
$$\ell=\max\bigl\{i\leqslant m:\ \rho^\tau(i)\notin\bigl\{\rho(1),\rho(2),\dots,\rho(n)\bigr\}\bigr\}<m,$$

and

$$\eta_i=\begin{cases}0 & \text{if } \rho^\tau(i)\in\{\rho(1),\rho(2),\dots,\rho(n)\},\\ 1 & \text{if } \rho^\tau(i)\notin\{\rho(1),\rho(2),\dots,\rho(n)\}\end{cases}\quad \text{for } i=k,k+1,\dots,\ell.$$

By the choice of m, we have $\rho^\tau(l+1)=\rho(j)$ for some $1\leqslant j\leqslant n$. Hence for $k\leqslant i\leqslant l$, we have

$$\tau|a_{\rho(n)}|\leqslant\tau|a_{\rho^\tau(l+1)}|\leqslant|a_{\rho^\tau(i)}|\leqslant\frac{1}{\tau}|a_{\rho^\tau(k)}|\leqslant\frac{1}{\tau}|a_{\rho(n)}|,$$

so

$$\frac{|a_{\rho(n)}|}{\tau}\leqslant\frac{|a_{\rho^\tau(i)}|}{\tau^2}\leqslant\frac{1}{\tau^2}\frac{|a_{\rho(n)}|}{\tau}.$$

Thus, using property $P(\tau)$ for the second inequality, we have

$$\begin{aligned}\left\|\sum_{i=k}^{l}\eta_i a_{\rho^\tau(i)}e_{\rho^\tau(i)}\right\| &\leqslant \frac{|a_{\rho(n)}|}{\tau}\max_{\pm}\left\|\sum_{i=k}^{l}\pm e_{\rho^\tau(i)}\right\| \\ &\leqslant \frac{C}{\tau^2}\left\|\sum_{i=k}^{l}a_{\rho^\tau(i)}e_{\rho^\tau(i)}\right\| \\ &= \frac{C}{\tau^2}\left\|\mathcal{G}_l^\tau(x)-\mathcal{G}_{k-1}^\tau(x)\right\| \\ &\leqslant \frac{2KC}{\tau^2}\|x\|.\end{aligned}$$

Thus,

$$\left\|\mathcal{G}_n(x)\right\| \leqslant \left\|\mathcal{G}_m^\tau(x)\right\| + \left\|\sum_{i=k}^{l}\eta_i a_{\rho^\tau(i)}e_{\rho^\tau(i)}\right\| \leqslant K\left(1+\frac{2C}{\tau^2}\right)\|x\|. \qquad \square$$

The last result has a converse.

Proposition 4.5. *Suppose that (e_i) is QG. Then (e_i) has $P(\tau)$ for all $0<\tau<1$ with a uniform constant. Moreover, for every branch greedy algorithm, we have $\|\mathcal{G}_n^\tau(x)\| \leqslant K(\tau)\|x\|$ for all $n \geqslant 1$ and $x \in X$.*

Proof. It is proved in [2, pp. 70–71] that if (e_i) is quasi-greedy with constant K and $(a_i)_{i\in A}$ are scalars such that $|a_i| \geqslant 1$ then

$$\max_{\pm}\left\|\sum_{i\in A}\pm e_i\right\| \leqslant 4K^2\left\|\sum_{i\in A}a_i e_i\right\|.$$

Hence (e_i) has $P(\tau)$ for all $0<\tau<1$ with uniform constant $4K^2$. The second assertion follows the fact that weak thresholding with respect to a quasi-greedy basis is bounded with constant $K(\tau)$ depending on the weakness parameter τ and the quasi-greedy constant [9, pp. 312–314]. $\square$

Next we give an application of Proposition 4.4 to infinite-dimensional spaces.

Corollary 4.6. *Suppose that $(x_i)_{i=1}^\infty$ is a weakly null semi-normalized BQG(τ) basic sequence in an infinite-dimensional Banach space. Then (x_i) has a quasi-greedy subsequence.*

Proof. It follows from Elton's partial unconditionality theorem [6] that there exist a subsequence (y_i) and a constant $K(\tau)$ such that for all finite sets $E \subset \mathbb{N}$ and scalars $(a_i)_{i\in E}$ satisfying $1 \leqslant |a_i| \leqslant 1/\tau^2$, we have

$$\left\|\sum_{i\in E}\pm a_i y_i\right\| \leqslant K(\tau)\left\|\sum_{i\in E}a_i y_i\right\|.$$

By convexity,

$$\Big\|\sum_{i\in E}\pm y_i\Big\| \leqslant \max_{\pm}\Big\|\sum_{i\in E}\pm a_i y_i\Big\| \leqslant K(\tau)\Big\|\sum_{i\in E}a_i y_i\Big\|.$$

Hence $(y_i)_{i=1}^{\infty}$ is $BQG(\tau)$ and has property $P(\tau)$ with constant $K(\tau)$. Thus, (y_i) is quasi-greedy by Proposition 4.4. $\square$

Remark 4.7. It is an open question whether or not every semi-normalized weakly null sequence (x_i) in a Banach space has a quasi-greedy subsequence (see [5]). (The answer is positive if the Banach space does not have c_0 as a spreading model [2, Corollary 56].) By Corollary 4.6, whether or not (x_i) has a $BQG(\tau)$ subsequence is an equivalent question.

5. Branch greedy systems

Definition 5.1. Let $0<\tau<1$. We say that (e_i) is *branch greedy* with weakness parameter τ ($BG(\tau)$) and constant K if for all $x\in X$ and $0\leqslant k\leqslant N$, we have

$$\big\|x-\mathcal{G}_k^\tau(x)\big\| \leqslant K\sigma_k(x),$$

where

$$\sigma_k(x)=\min\bigg\{\Big\|x-\sum_{i\in A}a_ie_i\Big\|:\ A\subset\{1,\dots,N\},\ |A|\leqslant k\bigg\}.$$

Theorem 5.2. *Suppose that $(e_i)_{i=1}^N$ is $BG(\tau)$ with constant K. Then (e_i) is K-unconditional and democratic with constant $K(1+\frac{1}{\tau})$.*

Proof. To show unconditionality, suppose that $x=\sum_{i\in A}a_ie_i$ and that $B\subset A$. Let $r:=|A\setminus B|$. Consider $y=\sum_{i\in B}a_ie_i+M\sum_{i\in A\setminus B}e_i$, where $M>\frac{1}{\tau}\max_{i\in B}|a_i|$. Clearly $\mathcal{G}_r^\tau(x)=M\sum_{i\in A\setminus B}e_i$, and since $x-y$ is an r-term approximation to y we have

$$\Big\|\sum_{i\in B}a_ie_i\Big\| \leqslant K\sigma_r(y)\leqslant K\big\|y-(y-x)\big\|=K\|x\|.$$

Thus, (e_i) is K-unconditional. To show that (e_i) is $K(1+\frac{1}{\tau})$-democratic, let A,B satisfy $|B|\leqslant|A|:=n$. Consider $x=\theta\sum_{i\in B\setminus A}e_i+\sum_{i\in A}e_i$, where $0<\theta<\tau$. Then $G_n^\tau(x)=\sum_{i\in A}e_i$, so

$$\theta\Big\|\sum_{i\in B\setminus A}e_i\Big\| \leqslant K\sigma_n(x)\leqslant K\Big\|\sum_{i\in A}e_i\Big\|,$$

where the second inequality follows from the fact that $|B\setminus A|\leqslant n$. By unconditionality, $\|\sum_{i\in A\cap B}e_i\|\leqslant K\|\sum_{i\in A}e_i\|$. Hence

$$\Big\|\sum_{i\in B}e_i\Big\| \leqslant \Big\|\sum_{i\in B\setminus A}e_i\Big\|+\Big\|\sum_{i\in A\cap B}e_i\Big\| \leqslant K\bigg(1+\frac{1}{\theta}\bigg)\Big\|\sum_{i\in A}e_i\Big\|.$$

Since $\theta<\tau$ is arbitrary, we get that (e_i) is democratic with constant $K(1+\frac{1}{\tau})$. $\square$

Corollary 5.3. *If (e_i) is $BG(\tau)$ with constant K then (e_i) is greedy with constant $K + K^4(1+\frac{1}{\tau})$.*

Proof. This follows from the result of Konyagin and Temlyakov that a basis is greedy if and only if it is unconditional and democratic [8, Theorem 1] and from the estimate of the greedy basis constant given in [16, Theorem 1] (see also [1]). □

6. Branch almost greedy systems

Definition 6.1. We say that (e_i) is *branch almost greedy* with weakness parameter τ ($BAG(\tau)$) and constant K if for all $x \in X$ and $0 \leqslant k \leqslant N$, we have

$$\left\|x - \mathcal{G}_k^\tau(x)\right\| \leqslant K\tilde{\sigma}_k(x),$$

where

$$\tilde{\sigma}_k(x) = \min\left\{\left\|x - P_A(x)\right\|:\ A \subset \{1, \dots, N\}, |A| \leqslant k\right\}.$$

Recall that the *fundamental function* (φ_n) of (e_i) is defined by

$$\varphi(n) = \sup\left\{\left\|\sum_{i\in A} e_i\right\|:\ |A| \leqslant n\right\}.$$

Lemma 6.2. *Let $0 < \tau < 1$. Suppose that $(e_j)_{j=1}^N$ is $BQG(\tau)$ with constant K and democratic with constant Δ. Suppose also that (8) is satisfied with constant C for all A with $|A| \leqslant N/2$ and for all scalars $(a_i)_{i\in A}$, with $1 \leqslant |a_i| \leqslant 1/\tau^2$. Then $(e_j)_{j=1}^N$ has property $P(\tau)$ with constant $6KC\Delta$.*

Proof. Suppose that $|A| := k > N/2$ and that $x = \sum_{j\in A} a_j e_j$ where $1 \leqslant |a_j| \leqslant \frac{1}{\tau^2}$ ($j \in A$). Let $\mathcal{G}_{[N/2]}^\tau(x) := \sum_{j\in B} a_j e_j$. Then

$$\left\|\sum_{j\in B} a_j e_j\right\| = \left\|\mathcal{G}_{[N/2]}^\tau(x)\right\| \leqslant K\|x\|.$$

Since $|B| = [N/2]$, we have by assumption that

$$\begin{aligned} C\left\|\sum_{j\in B} a_j e_j\right\| &\geqslant \left\|\sum_{j\in B} e_j\right\| \\ &\geqslant \frac{\varphi([N/2])}{\Delta} \\ &\geqslant \frac{\varphi(k)}{3\Delta} \geqslant \frac{1}{6\Delta}\max_{\pm}\left\|\sum_{j\in A} \pm e_j\right\|. \qquad \square \end{aligned}$$

Remark 6.3. Note that the proof only requires the democratic condition for sets of cardinality at most $N/2$, i.e., that if $|E| \leqslant |F| \leqslant N/2$ then $\|\sum_{i\in E} e_i\| \leqslant \Delta\|\sum_{i\in F} e_i\|$. This observation will be needed in the proof of Theorem 6.4 below.

Theorem 6.4. *Suppose that $(e_i)_{i=1}^N$ is BAG(τ) with constant K. Then $(e_i)_{i=1}^N$ is democratic with constant $3(K^2(1+K)/\tau^2)(1+(K/\tau))$ and QG with constant $(1+K)(1+12K^5(1+(K/\tau))/\tau^6)$.*

Proof. First observe that for all $x \in X$ and $k \geqslant 0$, we have

$$\left\|\mathcal{G}_k^\tau(x)\right\| \leqslant \left\|x - \mathcal{G}_k^\tau(x)\right\| + \|x\| \leqslant K\tilde{\sigma}_k(x) + \|x\| \leqslant (K+1)\|x\|.$$

Hence $(e_i)_{i=1}^N$ is $BQG(\tau)$ with constant $K+1$. Next we prove that (8) is satisfied with constant $\frac{K^2}{\tau^2}$ for all A with $|A| \leqslant N/2$ and for all scalars $(a_i)_{i\in A}$, with $1 \leqslant |a_i| \leqslant 1/\tau^2$. Suppose that $n := |A| \leqslant N/2$ and that $|a_i| \geqslant 1$ $(i \in A)$. Choose $D \subset \{1,\dots,N\}$ such that A and D are disjoint and $|D| = |A|$. Consider $x = \theta\sum_{i\in D} e_i + \sum_{i\in A} a_i e_i$, where $0 < \theta < \tau$. Then $\mathcal{G}_n^\tau(x) = \sum_{i\in A} a_i e_i$, and hence

$$\theta\left\|\sum_{i\in D} e_i\right\| \leqslant K\tilde{\sigma}_n(x) \leqslant K\left\|\sum_{i\in A} a_i e_i\right\|. \tag{9}$$

(For future reference, note that (9) is valid provided $|D| \leqslant |A|$ and $D \cap A = \emptyset$.) Now consider $y = \sum_{i\in D} e_i + \theta\sum_{i\in A} \pm e_i$. Then $\mathcal{G}_n^\tau(y) = \sum_{i\in D} e_i$, and hence

$$\theta\left\|\sum_{i\in A} \pm e_i\right\| \leqslant K\tilde{\sigma}_n(y) \leqslant K\left\|\sum_{i\in D} e_i\right\|.$$

Combining these estimates, and letting $\theta \downarrow \tau$, we get

$$\max_{\pm}\left\|\sum_{i\in A} \pm e_i\right\| \leqslant \frac{K^2}{\tau^2}\left\|\sum_{i\in A} a_i e_i\right\|. \tag{10}$$

Next we prove that $(e_i)_{i=1}^N$ is democratic. First suppose that $|B| \leqslant n := |A| \leqslant N/2$. Using (9) with A replaced by $A \setminus B$ and D replaced by $B \setminus A$ (noting that $|B \setminus A| \leqslant |A \setminus B|$) for the first inequality, and (10) for the third inequality, we get

$$\begin{aligned}\left\|\sum_{i\in B\setminus A} e_i\right\| &\leqslant \frac{K}{\tau}\left\|\sum_{i\in A\setminus B} e_i\right\| \\ &\leqslant \frac{K}{\tau}\max_{\pm}\left\|\sum_{i\in A} \pm e_i\right\| \leqslant \frac{K^3}{\tau^3}\left\|\sum_{i\in A} e_i\right\|.\end{aligned}$$

Similarly,

$$\left\|\sum_{i\in A\cap B} e_i\right\| \leqslant \max_{\pm}\left\|\sum_{i\in A} \pm e_i\right\| \leqslant \frac{K^2}{\tau^2}\left\|\sum_{i\in A} e_i\right\|.$$

Thus,

$$\left\|\sum_{i\in B} e_i\right\| \leqslant \left\|\sum_{i\in B\setminus A} e_i\right\| + \left\|\sum_{i\in A\cap B} e_i\right\| \leqslant \frac{K^2}{\tau^2}\left(1+\frac{K}{\tau}\right)\left\|\sum_{i\in A} e_i\right\|,$$

and hence

$$\varphi(n) \leqslant \frac{K^2}{\tau^2}\left(1+\frac{K}{\tau}\right)\left\|\sum_{i\in A} e_i\right\|.$$

Hence from Lemma 6.2 and Remark 6.3 we deduce that (e_i) has property $P(\tau)$ with constant $(6K^5/\tau^4)(1+K/\tau)$, and then it follows from Proposition 4.4 that (e_i) is QG with constant $(1+K)(1+(12K^5/\tau^6)(1+K/\tau))$.

To complete the proof that (e_i) is democratic, suppose that $n > N/2$ and that $|A| = n$. There exists $B \subset A$ with $|B| = [N/2]$ such that $\mathcal{G}^{\tau}_{[N/2]}(\sum_{i\in A} e_i) = \sum_{i\in B} e_i$. Then $\|\sum_{i\in B} e_i\| \leqslant (1+K)\|\sum_{i\in A} e_i\|$ and hence

$$\begin{aligned}\varphi(n) \leqslant 3\varphi\big([N/2]\big) &\leqslant 3\frac{K^2}{\tau^2}\left(1+\frac{K}{\tau}\right)\left\|\sum_{i\in B} e_i\right\| \\ &\leqslant 3\frac{K^2(1+K)}{\tau^2}\left(1+\frac{K}{\tau}\right)\left\|\sum_{i\in A} e_i\right\|.\end{aligned}$$

This proves that (e_i) is democratic with constant

$$3\frac{K^2(1+K)}{\tau^2}\left(1+\frac{K}{\tau}\right). \qquad \square$$

Corollary 6.5. *If $(e_i)_{i=1}^N$ is BAG(τ) then $(e_i)_{i=1}^N$ is AG with AG constant depending only on τ and the BAG constant of $(e_i)_{i=1}^N$.*

Proof. This follows from the result [3, Theorem 33] that a quasi-greedy and democratic system is almost greedy with constant depending only on the quasi-greedy and democratic constants of the system. $\square$

7. Branch semi-greedy systems

For $x = \sum_{i=1}^N a_i e_i$ and $1 \leqslant n \leqslant N$, let us say that $\Lambda^\tau(x,n) \subseteq \{1,\dots,N\}$ is a *weak thresholding set with weakness parameter* τ if $|\Lambda^\tau(n,x)| = n$ and

$$\min\big\{|a_i|:\ i \in \Lambda^\tau(x,n)\big\} \geqslant \tau \max\big\{|a_i|:\ i \in \{1,\dots,N\} \setminus \Lambda^\tau(x,n)\big\}.$$

We begin with a weak thresholding version of [2, Theorem 32]. We omit the proof as only minor changes to the proof given in [2] are required.

Theorem 7.1. *Let $(e_i)_{i=1}^N$ be an AG system with QG constant K and democratic constant Δ. Then, for all $x \in X$ and weak thresholding sets $\Lambda^\tau(x,n)$, there exist scalars c_i ($i \in \Lambda^\tau(x,n)$) such that*

$$\left\| x - \sum_{i \in \Lambda^\tau(x,n)} c_i e_i \right\| \leqslant \big(1 + 3K + 16K^2\Delta/\tau\big)\sigma_n(x).$$

Combining Theorem 6.4 and Corollary 6.5 yields the following.

Corollary 7.2. *Suppose that $(e_i)_{i=1}^N$ is $BAG(\tau)$ with constant K. Then there exists a constant $C(K,\tau)$ such that for all $x \in X$ and weak thresholding sets $\Lambda^\tau(x,n)$, there exist scalars c_i ($i \in \Lambda^\tau(x,n)$) such that*

$$\left\| x - \sum_{i \in \Lambda^\tau(x,n)} c_i e_i \right\| \leqslant C(K,\tau)\sigma_n(x).$$

The following definition generalizes the notion of semi-greedy basis introduced in [2, Section 3].

Definition 7.3. We say that (e_i) is *branch semi-greedy* with weakness parameter τ ($BSG(\tau)$) and constant K if for all $x \in X$ and $0 \leqslant k \leqslant N$, there exist scalars $c_1, \dots, c_k$ such that

$$\left\| x - \sum_{i=1}^k c_i e_{\rho_x^\tau(i)} \right\| \leqslant K\sigma_k(x).$$

The remainder of this section provides a partial answer to the following open question: does the $BSG(\tau)$ property imply the $BAG(\tau)$ property? The converse is true; in fact, by Corollary 7.2, if the system is $AG(\tau)$ then *every* branch satisfies the $BSG(\tau)$ condition.

Our results involve the *basis constant* of $(e_i)_{i=1}^N$, denoted β, which is defined as follows:

$$\beta := \max\left\{ \left\| \sum_{i=1}^k e_i^*(x) e_i \right\| :\ \|x\| = 1, 1 \leqslant k \leqslant N \right\}.$$

First we show that a $BSG(\tau)$ *Schauder basis* is *superdemocratic*.

Theorem 7.4. *Let $(e_i)_{i=1}^N$ be a $BSG(\tau)$ with constant K. Then $(e_i)_{i=1}^N$ is superdemocratic, i.e. there exists a constant $C > 0$ (depending only on K, τ, and β) such that for all $D \subseteq \{1, \dots, N\}$, we have*

$$\varphi\big(|D|\big) \leqslant C \min_{\pm} \left\| \sum_{i \in D} \pm e_i \right\|.$$

Proof. We assume for convenience that N is even. Suppose that $A \subseteq \{1, \dots, N/2\}$ and $B \subseteq \{N/2+1, \dots, N\}$ with $|A| = |B| := k$. For any choice of signs, consider

$$x := \frac{\tau}{2} \sum_{i \in A} \pm e_i + \sum_{i \in B} \pm e_i.$$

Since $(e_i)_{i=1}^N$ is $BSG(\tau)$ there exist scalars c_i $(i \in B)$ such that

$$\left\| \frac{\tau}{2} \sum_{i \in A} \pm e_i + \sum_{i \in B} c_i e_i \right\| \leqslant K \sigma_k(x),$$

where K is the $BSG(\tau)$ constant. Hence

$$\left\| \sum_{i \in A} \pm e_i \right\| \leqslant \frac{2K\beta}{\tau} \sigma_k(x) \leqslant \frac{2K\beta}{\tau} \left\| \sum_{i \in B} \pm e_i \right\|.$$

Similarly,

$$\left\| \sum_{i \in B} \pm e_i \right\| \leqslant \frac{2K(\beta+1)}{\tau} \left\| \sum_{i \in A} \pm e_i \right\|.$$

Combining these inequalities, we get

$$\left\| \sum_{i \in A} e_i \right\| \leqslant \frac{4K^2\beta(1+\beta)}{\tau^2} \left\| \sum_{i \in A} \pm e_i \right\|$$

and

$$\left\| \sum_{i \in B} e_i \right\| \leqslant \frac{4K^2\beta(1+\beta)}{\tau^2} \left\| \sum_{i \in B} \pm e_i \right\|.$$

For $1 \leqslant k \leqslant N/2$, define

$$\psi(k) := \max\left\{ \left\| \sum_{i \in D} e_i \right\| \colon D \subset \{1, \dots, N/2\} \text{ or } D \subset \left\{ \frac{N}{2}+1, \dots, N \right\}, |D| \leqslant k \right\}.$$

By the triangle inequality, $\varphi(n) \leqslant 4\psi(n/2) \leqslant 4\varphi(n/2)$ for $1 \leqslant n \leqslant N$ provided n is even. From the above, we obtain

$$\left\| \sum_{i \in D} \pm e_i \right\| \geqslant \frac{\tau^2}{4K^2\beta(1+\beta)} \psi\bigl(|D|\bigr)$$

for all $D \subset \{1, \dots, N/2\}$ and $D \subset \{N/2+1, \dots, N\}$. For $D \subseteq \{1, \dots, N\}$, set $A := D \cap \{1, \dots, N/2\}$ and $B := D \cap \{N/2+1, \dots, N\}$. Then, provided $|D|$ is even, we obtain

$$\begin{aligned} \left\| \sum_{i \in D} \pm e_i \right\| &\geqslant \frac{1}{2(1+\beta)} \left(\left\| \sum_{i \in A} \pm e_i \right\| + \left\| \sum_{i \in B} \pm e_i \right\| \right) \\ &\geqslant \left(\frac{1}{2(1+\beta)} \right) \left(\frac{\tau^2}{4K^2\beta(1+\beta)} \right) \psi\bigl(|D|/2\bigr) \\ &\geqslant \frac{\tau^2}{8K^2\beta(1+\beta)^2} \frac{\varphi(|D|)}{4}. \qquad \square \end{aligned}$$

Minor adjustments to the previous proof yield the following stronger result which is needed below.

Proposition 7.5. *Let $(e_i)_{i=1}^N$ be BSG(τ) with constant K. There exists $C > 0$, depending only on K, τ, and β, such that*

$$\varphi\bigl(|D|\bigr) \min_{i \in D} |a_i| \leqslant C \biggl\| \sum_{i \in D} a_i e_i \biggr\|$$

for all $D \subset \{1, \dots, N\}$ and all scalars a_i $(i \in D)$. In particular, (e_i) has property $P(\tau)$ (with constant depending only on K, τ, and β).

The greedy approximants have the semigroup property: $\mathcal{G}_m(\mathcal{G}_n(x)) = \mathcal{G}_m(x)$ for $m \leqslant n$ and $x \in X$. However, branch greedy approximants $\mathcal{G}_m^\tau(x)$ satisfying our conditions (a)–(c) need not have the semigroup property, and this complicates the proof of Theorem 7.7 below. The following lemma circumvents this difficulty.

Lemma 7.6. *Let $(e_i)_{i=1}^N$ be BSG(τ) with constant K. Suppose that $1 \leqslant n \leqslant N$ and that $n/2 \leqslant m \leqslant n$. Then for all $x \in X$, we have*

$$\bigl\| \mathcal{G}_m^\tau(x) \bigr\| \leqslant C \max_{m \leqslant k \leqslant n} \bigl\| \mathcal{G}_k^\tau\bigl(\mathcal{G}_n^\tau(x)\bigr) \bigr\|,$$

where C depends only on K, τ, and β.

Proof. Let $x := \sum_{i=1}^N a_i e_i$ have branch greedy ordering ρ^τ. Let k be the least integer such that $\operatorname{supp}(\mathcal{G}_k^\tau(\mathcal{G}_n^\tau(x))) \supseteq \operatorname{supp}(\mathcal{G}_m^\tau(x))$. Then $\mathcal{G}_k^\tau(\mathcal{G}_n^\tau(x)) = \sum_{i \in A} a_i e_i$ for some $A \subset \{1, \dots, N\}$. From the definition of k we get that

$$\min_{i \in A} |a_i| \geqslant \tau \min_{1 \leqslant i \leqslant m} |a_{\rho^\tau(i)}|. \tag{11}$$

For some $B \subset \{1, \dots, N\}$ we have

$$\mathcal{G}_k^\tau\bigl(\mathcal{G}_n^\tau(x)\bigr) = \mathcal{G}_m^\tau(x) + \sum_{i \in B} a_i e_i. \tag{12}$$

Note that by definition of the branch greedy ordering,

$$\tau \max_{i \in B} |a_i| \leqslant \min_{1 \leqslant i \leqslant m} \bigl|a_{\rho^\tau}(i)\bigr| \tag{13}$$

and that $|B| \leqslant n/2 \leqslant m \leqslant k$. Combining (11) and (13), we have that $\tau^2 \max_{i \in B} |a_i| \leqslant \min_{i \in A} |a_i|$. Hence, using Proposition 7.5, there exists $C_1(K, \tau, \beta)$ such that

$$\begin{aligned} \biggl\| \sum_{i \in B} a_i e_i \biggr\| &\leqslant 2\Bigl(\max_{i \in B} |a_i|\Bigr)\varphi\bigl(|B|\bigr) \\ &\leqslant \frac{2}{\tau^2}\Bigl(\min_{i \in A} |a_i|\Bigr)\varphi(k) \end{aligned}$$

$$\leqslant \frac{C_1}{\tau^2}\left\|\sum_{i\in A} a_i e_i\right\|$$
$$= \frac{C_1}{\tau^2}\left\|\mathcal{G}_k^\tau\left(\mathcal{G}_n^\tau(x)\right)\right\|.$$

Finally, (12) and the triangle inequality yield

$$\left\|\mathcal{G}_m^\tau(x)\right\| \leqslant \left(1+\frac{C_1}{\tau^2}\right)\left\|\mathcal{G}_k^\tau\left(\mathcal{G}_n^\tau(x)\right)\right\|. \qquad \square$$

Finally, we give a partial answer to the open question raised after Definition 7.3 for a $BSG(\tau)$ basis with estimates involving the *cotype q constant* of X. Let $2 \leqslant q < \infty$. The cotype q constant C_q of X is the smallest constant such that

$$\left(\sum_{j=1}^n \|x_j\|^q\right)^{\frac{1}{q}} \leqslant C_q\left(\mathrm{Ave}_{\varepsilon_j=\pm 1}\left\|\sum_{j=1}^n \varepsilon_j x_j\right\|^q\right)^{\frac{1}{q}} \tag{14}$$

for all $x_1,\dots,x_n \in X$ and $n \in \mathbb{N}$. We recall that an infinite-dimensional Banach space X has *finite cotype*, i.e., $C_q < \infty$ for some $q < \infty$, if and only if there exist $n \in \mathbb{N}$ and $\varepsilon > 0$ such that X does not contain a $(1+\varepsilon)$-isomorphic copy of ℓ_∞^n [10].

Theorem 7.7. *Let $2 \leqslant q < \infty$. Suppose that $(e_i)_{i=1}^N$ is $BSG(\tau)$ with constant K. Then $(e_i)_{i=1}^N$ is AG with constant depending only on K, τ, β, q, and the cotype q constant C_q of X.*

Proof. By the previous result the superdemocratic constant S depends only on K, β, and τ. It is shown in [3, Proposition 41] that the fundamental function of a superdemocratic basis has the *lower regularity property*, i.e. φ satisfies

$$\varphi(mn) \geqslant \frac{1}{S^2 C_q} m^{1/q}\varphi(n) \quad (m, n \geqslant 1). \tag{15}$$

We may assume that N is even. We shall not keep track of the constants, so C_1, C_2 etc., will denote constants depending only on K, τ, β, q, and C_q. Recall that a system is AG if and only if it is QG and democratic [3, Theorem 33]. Since (e_i) is superdemocratic, it suffices to show that (e_i) is QG. Let $F \subset \{1,\dots,N/2\}$ and let $n := |F|$. Consider a vector $x = \sum_{i\in F} a_i e_i$ with $\|x\| = 1$ and $\mathrm{supp}(x) \subset F$. Let ρ^τ be the branch greedy ordering for x and let $1 \leqslant k \leqslant n$. Note that $|e_i^*(x - \mathcal{G}_k^\tau(x))| \leqslant |a_{\rho^\tau(k)}|/\tau$ $(1 \leqslant i \leqslant N)$. Hence

$$\left\|x - \mathcal{G}_k^\tau(x)\right\| \leqslant \frac{2}{\tau}|a_{\rho^\tau(k)}|\varphi(n-k).$$

By Proposition 7.5,

$$\left\|\mathcal{G}_k^\tau(x)\right\| \geqslant \frac{1}{C_1}\Big(\min_{1\leqslant i\leqslant k}|a_{\rho^\tau(i)}|\Big)\varphi(k) \geqslant \frac{\tau}{C_1}|a_{\rho^\tau(k)}|\varphi(k).$$

Hence

$$\frac{\|\mathcal{G}_k^\tau(x)\| - 1}{\|\mathcal{G}_k^\tau(x)\|} \leqslant \frac{\|x - \mathcal{G}_k^\tau(x)\|}{\|\mathcal{G}_k^\tau(x)\|} \leqslant \frac{2C_1}{\tau^2}\frac{\varphi(n-k)}{\varphi(k)}.$$

By (15) the right-hand side tends to zero as $k/n \to 1$. Hence there exists $\alpha < 1$ (depending on K, τ, β, q, and C_q) such that

$$\left\|\mathcal{G}_k^\tau(x)\right\| \leqslant C_2 \quad \text{for all } k \geqslant \alpha n. \tag{16}$$

Now we iterate (16). Let $n_1 := [\alpha n]$ and suppose $k \geqslant \alpha^2 n$. By Lemma 7.6 and (16) we get

$$\begin{aligned}\left\|\mathcal{G}_k^\tau(x)\right\| &\leqslant \max_{\alpha^2 n \leqslant j \leqslant n_1} C_3 \left\|\mathcal{G}_j^\tau\left(\mathcal{G}_{n_1}^\tau(x)\right)\right\| \\ &\leqslant C_3 C_2 \left\|\mathcal{G}_{n_1}^\tau(x)\right\| \\ &\leqslant C_3 C_2^2.\end{aligned}$$

Clearly, we can continue iterating (16) in this way. Iterating m times, where $\alpha^m \leqslant 1/2$, we get

$$\left\|\mathcal{G}_k^\tau(x)\right\| \leqslant C_4 \quad \text{for all } k \geqslant n/2. \tag{17}$$

Fix $1 \leqslant k \leqslant n$. Let $A := \{\rho^\tau(1), \dots, \rho^\tau(k)\}$ and let $B := \{\rho^\tau(k+1), \dots, \rho^\tau(2k)\}$. Choose $D \subseteq \{N/2+1, \dots, N\}$ with $|D| = k$ and consider

$$y := \frac{\tau}{2}\sum_{i \in F\setminus A} a_i e_i + |a_{\rho^\tau(k)}|\left(\sum_{i\in D} e_i\right).$$

Then

$$\sigma_k(y) \leqslant \left\|\frac{\tau}{2}x + |a_{\rho^\tau(k)}|\left(\sum_{i\in D} e_i\right)\right\| \leqslant \frac{\tau}{2} + |a_{\rho^\tau(k)}|\varphi(k). \tag{18}$$

Since (e_i) is $BSG(\tau)$ and $D = \{\rho_y^\tau(i)\colon 1 \leqslant i \leqslant k\}$ there exist scalars c_i $(i \in D)$ such that

$$\left\|\frac{\tau}{2}\sum_{i\in F\setminus A} a_i e_i + \sum_{i\in D} c_i e_i\right\| \leqslant K\sigma_k(y). \tag{19}$$

(18) and (19) yield

$$\frac{\tau}{2}\left\|\sum_{i\in F\setminus A} a_i e_i\right\| \leqslant K\beta\left(\frac{\tau}{2} + |a_{\rho^\tau(k)}|\varphi(k)\right).$$

Hence

$$\left\|\mathcal{G}_k^\tau(x)\right\| = \left\|\sum_{i\in A} a_i e_i\right\| \leqslant 1 + K\beta\left(\frac{\tau}{2} + |a_{\rho^\tau(k)}|\varphi(k)\right). \tag{20}$$

Let $z := x - \sum_{i\in A} a_i e_i$. Then $\sigma_k(z) \leqslant \|x\| \leqslant 1$. Since (e_i) is $BSG(\tau)$ there exist scalars (c_i) $(i \in B)$ with $\|z - \sum_{i\in B} c_i e_i\| \leqslant K\sigma_k(z) \leqslant K$. Hence

$$\left\|\sum_{i\in A} a_i e_i + \sum_{i\in B} c_i e_i\right\| = \left\|x - \left(z - \sum_{i\in B} c_i e_i\right)\right\| \leqslant 1 + K. \tag{21}$$

Let $E := \{i \in B\colon |c_i| \geqslant \tau^2 |a_{\rho(k)}|\}$. Note that $E \supseteq \{i \in B\colon |c_i| \geqslant \tau \min_{1\leqslant j\leqslant k} |a_{\rho^\tau(j)}|\}$. Hence there exist $E_1 \subseteq E$ and m with $k \leqslant m \leqslant 2k$ such that

$$\sum_{i\in A} a_i e_i + \sum_{i\in E_1} c_i e_i = \mathcal{G}_m^\tau\left(\sum_{i\in A} a_i e_i + \sum_{i\in B} c_i e_i\right).$$

So (17) and (21) yield

$$\left\|\sum_{i\in A} a_i e_i + \sum_{i\in E_1} c_i e_i\right\| \leqslant C_4(1 + K). \tag{22}$$

On the other hand, Proposition 7.5 yields

$$\tau^2 |a_{\rho^\tau(k)}| \varphi(k) \leqslant C_1 \left\|\sum_{i\in A} a_i e_i + \sum_{i\in E_1} c_i e_i\right\|. \tag{23}$$

Combining (20), (22), and (23), we get $\|\mathcal{G}_k^\tau(x)\| \leqslant C_5$. By Proposition 7.5 again, (e_i) has property $P(\tau)$. Thus, by the proof of Proposition 4.4, we get that the greedy approximants $\mathcal{G}_k(x)$ satisfy $\|\mathcal{G}_k(x)\| \leqslant C_6$.

Similarly, we get $\|\mathcal{G}_k(x')\| \leqslant C_5$ for all x' of the form $x' = \sum_{i\in F'} a_i e_i$, where $F' \subset \{N/2 + 1, \dots, N\}$.

Finally, consider $x = \sum_{i=1}^N a_i e_i$ and set $x = y + z$, where $y = \sum_{i=1}^{N/2} a_i e_i$ and $z = \sum_{i=N/2+1}^N a_i e_i$. Then $\mathcal{G}_k(x) = \mathcal{G}_{k_1}(y) + \mathcal{G}_{k_2}(z)$, for some k_1, k_2 with $k = k_1 + k_2$. Thus,

$$\begin{aligned}\|\mathcal{G}_k(x)\| &\leqslant \|\mathcal{G}_{k_1}(y)\| + \|\mathcal{G}_{k_2}(z)\| \\ &\leqslant C_6\big(\|y\| + \|z\|\big) \\ &\leqslant C_6(1 + 2\beta)\|x\|.\end{aligned}$$

Thus, $(e_i)_{i=1}^N$ is greedy with constant $C_6(1 + 2\beta)$. □

Combining Theorem 7.2 and Theorem 7.7 yields the following.

Corollary 7.8. *Let (e_n) be a Schauder basis for a Banach space of finite cotype. Then (e_n) is semi-greedy if and only if (e_n) is $BSG(\tau)$.*

References

[1] F. Albiac, P. Wojtaszczyk, Characterization of 1-greedy bases, J. Approx. Theory 138 (1) (2006) 65–86.

[2] S.J. Dilworth, N.J. Kalton, Denka Kutzarova, On the existence of almost greedy bases in Banach spaces, Studia Math. 159 (2003) 67–101.

[3] S.J. Dilworth, Denka Kutzarova, N.J. Kalton, V.N. Temlyakov, The thresholding greedy algorithm, greedy bases, and duality, Constr. Approx. 19 (2003) 575–597.

[4] S.J. Dilworth, Denka Kutzarova, P. Wojtaszczyk, On approximate ℓ_1 systems in Banach spaces, J. Approx. Theory 114 (2002) 214–241.

[5] S.J. Dilworth, E. Odell, Th. Schlumprecht, András Zsák, Partial unconditionality, Houston J. Math. 35 (2009) 1251–1311.

[6] John Elton, Weakly null normalized sequences in Banach spaces, PhD thesis, Yale University, 1978.

[7] Smbat Gogyan, On convergence of weak thresholding greedy algorithm in $L^1(0,1)$, J. Approx. Theory 161 (2009) 49–64.

[8] S.V. Konyagin, V.N. Temlyakov, A remark on greedy approximation in Banach spaces, East J. Approx. 5 (1999) 1–15.

[9] S.V. Konyagin, V.N. Temlyakov, Greedy approximation with regard to bases and general minimal systems, Serdica Math. J. 28 (2002) 305–328.

[10] B. Maurey, G. Pisier, Séries de variables aléatoires vectorielles indépendantes et propriétés geométriques des espaces de Banach, Studia Math. 58 (1976) 45–90.

[11] V.N. Temlyakov, Greedy approximation and m-term trigonometric approximation, Constr. Approx. 14 (1998) 569–587.

[12] V.N. Temlyakov, The best m-term approximation and greedy algorithms, Adv. Comput. Math. 8 (1998) 249–265.

[13] V.N. Temlyakov, Greedy approximations with regard to bases, in: Proc. Int. Conf. of Mathem., Madrid, August 22–30, 2006, European Math. Soc., 2007, pp. 1479–1504.

[14] V.N. Temlyakov, Greedy Approximation, Cambridge University Press, 2011.

[15] P. Wojtaszczyk, Greedy algorithm for general biorthogonal systems, J. Approx. Theory 107 (2000) 293–314.

[16] P. Wojtaszczyk, Greedy type bases in Banach spaces, in: Constructive Theory of Functions, DARBA, Sofia, 2003, pp. 136–155.