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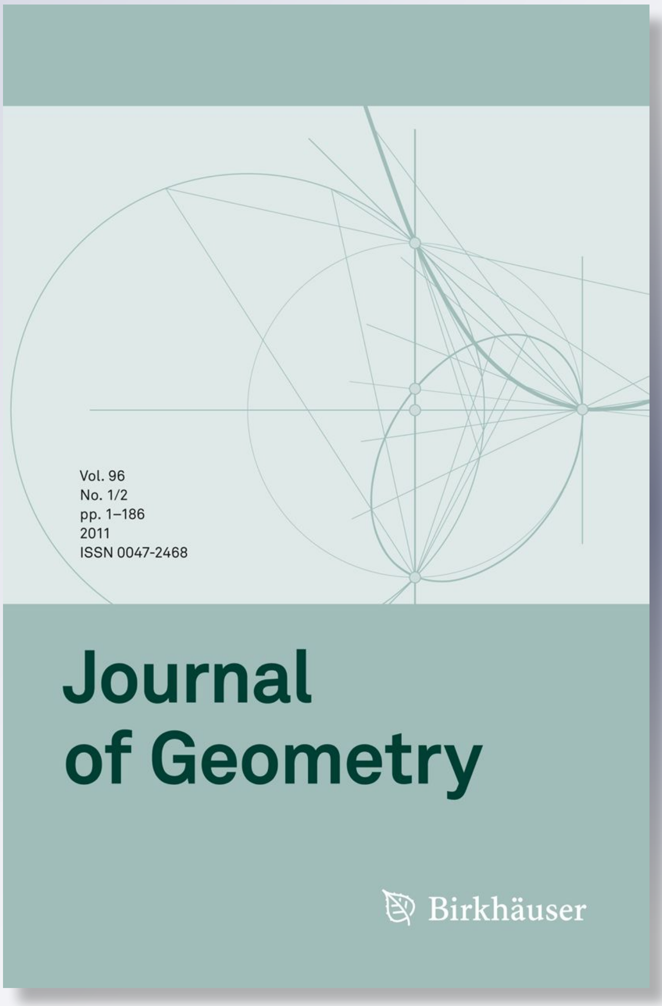

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# On a problem of Croft on optimally nested regular polygons

S. J. Dilworth and S. R. Mane

*To Hallard Croft, teacher and friend*

**Abstract.** We present a solution for the largest regular $m$-gon contained in a regular $n$-gon. We find that the answer depends critically on the coprimality of $m$ and $n$. We show that the optimal polygons are concentric if and only if $\gcd(m, n) > 1$. Our principal result is a complete solution for the case where $m$ and $n$ share a common divisor. For the case of coprime $m$ and $n$, we present partial results and a conjecture for the general solution. Our findings subsume some special cases which have previously been published on this problem.

**Mathematics Subject Classification (2010).** Primary 51M20; Secondary 52C15.

**Keywords.** Polygon containment, optimally nested polygons, axes of symmetry.

## 1. Introduction

In this paper we present solutions, and some conjectures, for the largest regular $m$-gon contained in a regular $n$-gon. The problem was posed to us by Dr. Hallard Croft (retired from the University of Cambridge). The problem posed by Croft is this: "$A$ and $B$ are regular, coplanar polygons, with respectively $m, n$ sides. $A$ lies inside (closed) $B$. The name of the game is to maximize (area $A$)/(area $B$). What can one say about the configurations that give a (global) maximum? The cases $m|n$ and $n|m$ are trivial."

The stated problem is a special case of the so-called "Polygon Containment Problem" which is important in computational mathematics and has important applications in industry (see e.g. [1]). In its full generality, the polygon containment problem pertains to irregular polygons, not necessarily convex,

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and can only be solved by numerical algorithms, and most of the analysis in [1] pertains to the complexity analysis of various algorithms. Our results below, however, are purely analytical, although admittedly we study only the special case of regular polygons. We also note in passing some results for regular polygons which give a higher packing density than circles in a plane (see Remark 4.10).

We summarize our results as follows. Detailed proofs are given in subsequent sections. We find that the answer depends crucially on the coprimality of $m$ and $n$. The optimal polygons are concentric if and only if $k = \gcd(m, n) > 1$. We present a complete answer to the problem when $\gcd(m, n) > 1$ (concentric polygons). Our results for $k = 1$ (coprime $(m, n)$, nonconcentric polygons) are more limited. We have results for $m = 3$ (triangle in polygon), $n = 3$ (polygon in triangle) and $m = 4$ (square in polygon). The case $n = 4$ (polygon in square) was solved by Dureisseix [3]. (Note that Dureisseix actually found the solution in 1997, in an unpublished note.) We also present a conjecture for the general solution for the case of coprime $(m, n)$ (nonconcentric polygons).

Special cases of the above problem have appeared as puzzles in various places, for example by Martin Gardner in *Scientific American*, specifically for the case of a square or rectangle in a triangle. The two cases $m = n \pm 1$ were solved by Daley [2]. Note that Daley [2] also solved the case of $m = n \pm 2$ for even $m$ and $n$; his results in this case are special cases of our general solution for arbitrary non-coprime $m$ and $n$.

## 2. General remarks

An equivalent formulation of the problem is to fix $B$, and to ask: for given fixed $n$, what is the polygon $A$ which maximizes $\triangle A/\triangle B$, for $m = 3, 4, 5, \ldots$? We have also seen papers on polygon containment which view the matter using the opposing but equivalent way: fix the inner polygon and determine the *smallest* enclosing polygon (this also maximizes the ratio of areas). We adopt the former viewpoint in our calculations. Some preliminary definitions are also required. For a regular polygon, a line joining the centre to a vertex is called a *radius* and a line joining the centre to the midpoint of an edge is called an *apothem*. Without loss of generality, we may fix $B$ to have unit radius. Let the radius of $A$ be $\rho_{mn}$. Then the ratio of areas is

$$\frac{\triangle A}{\triangle B} \equiv R_{mn} = \rho_{mn}^2 \, \frac{m}{n} \, \frac{\sin(2\pi/m)}{\sin(2\pi/n)}. \tag{2.1}$$

Hence we must determine the maximal radius of $A$. The solution must then specify the following: (i) the location of the centre of $A$, (ii) the orientation of $A$, and (iii) the value of the radius of $A$. It is obvious that in the limit $m \to \infty$ (for fixed $n$) the polygon $A$ approaches the inscribed circle of $B$ and the limiting radius and ratio of areas is given by

$$\rho_{\infty n} \equiv \lim_{m\to\infty} \rho_{mn} = \cos\frac{\pi}{n}, \qquad R_{\infty n} \equiv \lim_{m\to\infty} R_{mn} = \frac{\pi}{n}\cot\frac{\pi}{n}. \tag{2.2}$$

There are also obvious expressions for the limit $n \to \infty$ for fixed $m$, which we leave as an exercise for the reader.

## 3. Summary of results

We summarize our principal results below. The proofs of items 1–7 will be given in Sect. 4 below, and the proofs of individual special cases (8, 9 and 10) will be given in Sects. 5–7, respectively.

1. $A$ and $B$ are concentric iff $k = \gcd(m, n) > 1$.
2. $A$ and $B$ are concentric and share a common vertex if and only if $m|n$. $A$ and $B$ are concentric and share an edge of $A$ if and only if $n|m$.
3. Consider concentric $A$ and $B$ (i.e. $k > 1$). If $m/k$ is odd and $n/k$ is even then the optimal configurations are "radius to radius." If $m/k$ is even and $n/k$ is odd then the optimal configurations are "apothem to apothem." Both $m/k$ and $n/k$ are odd if and only if the optimal configurations are simultaneously "radius to radius" and "apothem to apothem." These terms will be explained below.
4. For $k > 1$ (concentric polygons), $A$ and $B$ share a common axis of symmetry. The number of common axes of symmetry is $k$.
5. If $m|n$ then $A$ and $B$ have $m$ points of contact, consisting of all the vertices of $A$. If $k > 1$ and $m$ does not divide $n$ then the number of points of contact is $2k$.
6. If (i) $m|n$ or (ii) $n$ is an odd multiple of $m/2$ (including $n = m/2$) then $A$ is inscribed in $B$. By "inscribed" we mean that every vertex of $A$ lies on the perimeter of $B$. If (i) $n|m$ or (ii) $m > n$ and $m$ is an odd multiple of $n/2$ then there is at least one, and at most two, point(s) of contact on every edge of $B$.
7. Also for concentric $A$ and $B$, the optimal radius of $A$ is (recall $B$ is normalized to a unit radius)

$$\rho_{mn} = \frac{\cos(\pi/n)}{\cos(\pi k/mn)}. \tag{3.1}$$

   Set $m = pn + q$, where $0 \le q < n$ and $p = 0, 1, 2, \ldots$ (such that $m \ge 3$). For fixed $n$ and $q$, the sequence $\{\rho_{pn+q,n}\}$ is monotone decreasing with limit $\cos(\pi/n)$ (see Eq. (2.2)). The ratio of areas is given by Eq. (2.1). For $q = 0$ (so $n|m$), the sequence $\{R_{pn+q,n}\}$ is monotone decreasing, but for $q > 0$, the sequence $\{R_{pn+q,n}\}$ is monotone increasing. In both cases, $\lim_{p\to\infty} R_{pn+q,n} = R_{\infty n}$ (see Eq. (2.2)).
8. $m = 3$: A maximal equilateral triangle $A$ in a regular polygon $B$ always shares a vertex in common with the enclosing polygon. $A$ and $B$ also share a common symmetry axis through the shared vertex.
9. $n = 3$: A maximal regular polygon $A$ inside an equilateral triangle $B$ always shares an edge in common with the enclosing triangle. The apothem of the shared edge is a common axis of symmetry of $A$ and $B$.

10. $m = 4$: A maximal square $A$ in a regular polygon $B$ shares a vertex in common with the enclosing polygon for the two cases $(m, n) = (4, 5)$ and $(4, 9)$ only, i.e. square in pentagon and square in nonagon. In these cases the square has three points of contact with the enclosing polygon. In all other cases, the square has four points of contact with the enclosing polygon, and the optimal configuration is "apothem to apothem." In all cases, $A$ and $B$ share a common axis of symmetry.
11. $n = 4$: Dureisseix [3] showed that for a maximal regular polygon $A$ inside a square $B$, the centre of the optimal polygon $A$ lies on a diagonal of the square $B$. That diagonal is a common axis of symmetry of $A$ and $B$. (Technically, we did not "prove" this, but we include his result for completeness.)
12. We present a conjecture for the general solution for the case of coprime $(m, n)$. Our conjecture rests on a key assumption, which we have not succeeded in proving.

## 4. Proofs of results in Sect. 3, nos. 1–7

In our derivations below, when we use the term 'subtend' the angle being subtended is always at the origin. We also define the term 'angle of incidence' as the angle between the radius of $A$ which makes contact with $B$ and the normal to the corresponding edge of $B$.

**Proposition 4.1.** *Suppose that $m$ and $n$ are coprime. Then there are no concentric optimal solutions.*

*Proof.* Suppose that $A$ and $B$ are concentric and optimal. First suppose that they do not have a common edge. Then all points of contact are vertices of $A$. Fix a vertex of $A$ that is a point of contact. Let $\theta > 0$ be the angle (measured counterclockwise) subtended by this point of contact with a vertex of $B$ on the same edge. Then no other point of contact can subtend an angle $\theta$ with a vertex of $B$, for otherwise $A$ and $B$ would have a nontrivial rotational symmetry. Since $A$ and $B$ are concentric, any other point of contact must subtend an angle $-\theta$ with a vertex of $B$, and there can be at most one such point of contact. Hence there are at most two points of contact. But then $A$ can clearly be displaced slightly (see Corollary 5.3 below) so as to have no points of contact, contradicting the optimality of $A$. Now suppose that $A$ and $B$ share an edge of $A$. Then the endpoints of this edge are vertices of $A$ that are symmetrically positioned around the midpoint of the edge of $B$. Since $A$ and $B$ are concentric there cannot be any other points of contact, for otherwise $A$ and $B$ would share a nontrivial rotational symmetry. Once again, we may displace $A$ slightly so that there are no points of contact, contradicting optimality. □

**Proposition 4.2.** *Suppose that $m$ and $n$ have greatest common divisor $k > 1$. Then all the optimal configurations are concentric.*

*Proof.* In this proof it is convenient to define $A$ and $B$ to be arbitrarily oriented (not necessarily optimal) regular polygons centred at the origin with $m$ and $n$ sides respectively. To derive a contradiction, suppose that $x + A \subset B$ is optimal for some nonzero $x \in \mathbb{R}^2$. Let $R$ be rotation through an angle of $2\pi/k$, which is a common symmetry of $A$ and $B$. Hence, for each $1 \leq i \leq k$, we have

$$R^i(x) + A = R^i(x + A) \subset R^i(B) = B.$$

So, by convexity,

$$A = \frac{1}{k}\sum_{i=1}^{k}(R^i(x) + A) \subset B.$$

Hence $A$ is also optimal. By convexity, $tx + A$ is optimal for all $0 \leq t \leq 1$. But if $A$ is optimal, then, for all sufficiently small $t > 0, tx + A$ will have points of contact in at most two edges of $B$ which are necessarily parallel to $x$. Clearly, $tx + A$ must touch at least two edges of $B$, for otherwise $tx + A$ can be displaced slightly so that it has no points of contact with $B$, which contradicts optimality of $tx + A$. Suppose that the only points of contact of $tx + A$ and $B$ are vertices of $tx + A$. Then, by choosing $t$ appropriately, we may assume that $tx + A$ and $B$ do not share any common vertices. But then $tx + A$ can be displaced slightly so that it has no points of contact with $B$ (see Corollary 5.3 below), contradicting maximality. It follows that $tx + A$ shares a common edge with $B$ for all sufficiently small $t > 0$. Hence $A$ shares a common edge with $B$ that is necessarily parallel to $x$. In particular, $m \geq n$. Since $A$ and $B$ are concentric the midpoint of the edge of $A$ coincides with the midpoint of the shared edge of $B$. Since there is at least one vertex of $A$ in any closed angular interval of length $2\pi/m$ and since $m \geq n$, it follows that on *every* edge of $B$ there are two vertices of $A$, where the corresponding radii of $A$ make angles of incidence $\pm\pi/m$ with the normal to that edge of $B$. Hence every edge of $B$ contains an edge of $A$, so $n|m$. Clearly, $tx + A$ will not be optimal in this case (in fact $tx + A$ will not be contained in $B$), which gives the desired contradiction. □

**Definition 4.3.** When $A$ and $B$ are concentric, let $O$ denote their common centre. To characterize the relative orientations of $A$ and $B$, we define "radius to radius" to mean a ray from $O$ through both a vertex of $A$ and a vertex of $B$, and we define "apothem to apothem" to mean a ray from $O$ through both the midpoint of an edge of $A$ and the midpoint of an edge of $B$. The terms "radius to apothem" and "apothem to radius" are similarly defined in the obvious way.

**Proposition 4.4.** *$A$ and $B$ are concentric and share a common vertex if and only if $m|n$. $A$ and $B$ are concentric and share an edge of $A$ if and only if $n|m$.*

*Proof.* Since $k > 1, A$ and $B$ are concentric. $A$ and $B$ can share a common vertex only if $m \leq n$. If $A$ and $B$ share a common vertex, the radius of $A$ equals the radius of $B$. It follows immediately that all the vertices of $A$ are also vertices of $B$, hence $m|n$. The converse is trivially true. This is a "radius–radius" configuration. Next, $A$ and $B$ can share an edge of $A$ only if $m \geq n$.

If $A$ and $B$ share an edge of $A$, then since $A$ and $B$ are concentric the midpoint of the edge of $A$ coincides with the midpoint of the shared edge of $B$ and the radii to the vertices of $A$ make angles of incidence $\pm\pi/m$ with the normal to the shared edge. Since there is at least one vertex of $A$ in any closed angular interval of length $2\pi/m$, it follows that there are two vertices of $A$ subtending angles of $\pm\pi/m$ around the midpoint of *every* edge of $B$, and these vertices must lie on the edges of $B$. Hence every edge of $B$ contains an edge of $A$, so $n|m$. The converse is trivially true. This is an "apothem–apothem" configuration. Clearly $A$ and $B$ are concentric and share a vertex and also an edge of $A$ if and only if $m = n$. □

**Theorem 4.5.** *Suppose that $m$ and $n$ have greatest common divisor $k > 1$. If $m/k$ is odd and $n/k$ is even then the optimal configurations are "radius to radius." If $m/k$ is even and $n/k$ is odd then the optimal configurations are "apothem to apothem." Both $m/k$ and $n/k$ are odd if and only if the optimal configurations are simultaneously "radius to radius" and "apothem to apothem."*

*Proof.* By Proposition 4.2 optimal configurations of $A$ and $B$ are concentric. If $A$ and $B$ have a common vertex or edge then the result follows from Proposition 4.4. Hence we may assume that all points of contact are vertices of $A$ that are interior points of edges of $B$. Suppose that one point of contact subtends an angle of $\theta > 0$ (measured counterclockwise) with a vertex of $B$ belonging to the same edge. If all points of contact subtend the same angle $\theta > 0$ with vertices of $B$ on their edges then we can rotate $A$ slightly so that it remains contained in $B$ but no longer has any points of contact. This would contradict the optimality of $A$. Hence there exists a second point of contact which does not subtend an angle of $\theta$. Since $A$ and $B$ are concentric it follows that this second point of contact subtends an angle of $-\theta$ with a vertex of $B$ on its edge. Hence there exist $0 \le r < m$ and $0 \le s < n$ such that

$$2\theta + \frac{2\pi r}{m} = \frac{2\pi s}{n}.$$

By repeatedly cycling through vertices of $A$, taken $r$ at a time, we find that for every odd number $\ell$ there exists a vertex of $A$ which subtends an angle of $\ell\theta$ with a vertex of $B$. Let $M := m/k$ and $N := n/k$. First consider the case where $M$ is odd. Note that

$$\theta = \left(\frac{-rn}{m} + s\right)\frac{\pi}{n} = \left(\frac{-rN}{M} + s\right)\frac{\pi}{n}.$$

Hence

$$M\theta = (-rN + sM)\,\frac{\pi}{n}.$$

Since $M$ is odd we can set $\ell := M$ and it follows that there exists a vertex of $A$ which subtends an angle of $(sM - rN)\pi/n$ with a vertex of $B$. Note that $sM - rN$ is an integer of undetermined parity. If $sM - rN$ is odd we get a "radius of $A$ to apothem of $B$" configuration. Such a configuration cannot be optimal since in this case $A$ is contained in the inscribed circle of $B$. Hence the optimal configuration occurs when $sM - rN$ is even. In this case the configuration is "radius to radius" as desired. If $M$ and $N$ are both odd then a "radius to radius" configuration is simultaneously "apothem to apothem" since

$$\left(1 + 2\frac{(M-1)}{2}\right)\frac{\pi}{m} = M\frac{\pi}{m} = N\frac{\pi}{n} = \left(1 + 2\frac{(N-1)}{2}\right)\frac{\pi}{n}.$$

Now consider the case where $M$ is even and $N$ is odd. Note that

$$\theta = \left(-r + \frac{sm}{n}\right)\frac{\pi}{m} = \left(-r + \frac{sM}{N}\right)\frac{\pi}{m}.$$

Hence

$$N\theta = (-rN + sM)\frac{\pi}{m}.$$

Since $N$ is odd we can set $\ell := N$ and it follows that there exists a vertex of $A$ which subtends an angle of $(-rN + sM)\pi/m$ with a vertex of $B$. If $-rN + sM$ is even we get a "radius to radius" configuration. But since $M$ is even and $N$ is odd, we have

$$\frac{M}{2}\left(\frac{2\pi}{m}\right) = N\left(\frac{\pi}{n}\right) = \frac{\pi}{n} + \frac{N-1}{2}\left(\frac{2\pi}{n}\right),$$

and hence this "radius to radius" configuration is simultaneously a "radius of $A$ to apothem of $B$" configuration. The latter configuration cannot be optimal since then $A$ would be contained in the inscribed circle of $B$. So the optimal configuration occurs when $-rN + sM$ is odd, which gives an "apothem of $A$ to radius of $B$" configuration. Since

$$\frac{M}{2}\left(\frac{2\pi}{m}\right) = \frac{\pi}{n} + \frac{N-1}{2}\left(\frac{2\pi}{n}\right),$$

it follows that this configuration is at the same time "apothem to apothem" as claimed.

It is now easily seen that in all cases the angle of incidence of the radius of $A$ which makes contact with $B$ and the normal to the corresponding edge of $B$ is $\pm\phi_{mn}$, where

$$\phi_{mn} = \frac{\pi k}{mn}. \tag{4.1}$$

□

It is immediate that the optimal radius of $A$ is

$$\rho_{mn} = \frac{\cos(\pi/n)}{\cos\phi_{mn}} = \frac{\cos(\pi/n)}{\cos(\pi k/mn)}.$$

This is Eq. (3.1). The following corollary is also an immediate consequence of Eq. (4.1).

**Corollary 4.6.** *If $k > 1$ then $A$ and $B$ share a common axis of symmetry. The number of common axes of symmetry is $k$.*

**Corollary 4.7.** *If $m|n$ then $A$ and $B$ have $m$ points of contact, consisting of all the vertices of $A$. If $k > 1$ and $m$ does not divide $n$ then the number of points of contact is $2k$.*

*Proof.* The case $m|n$ is trivial, so assume $m$ does not divide $n$. The proof of Theorem 4.5 shows that the points of contact are of two types: those subtending an angle $\phi_{mn}$ (type $\phi_{mn}$) and those subtending an angle $-\phi_{mn}$ (type $-\phi_{mn}$) with a midpoint of an edge of $B$. Now rotation through $2\pi/k$ is a common symmetry of both polygons, so there are at least $k$ points of contact of each type. On the other hand, any two points of contact of the same type must subtend an angle at the centre that is a common multiple of $2\pi/m$ and $2\pi/n$, and hence is a multiple of $2\pi/k$. So there are exactly $k$ points of contact of type $\phi_{mn}$ and $k$ points of contact of type $-\phi_{mn}$. □

**Corollary 4.8.** (a) *If* (i) *$m|n$ or* (ii) *$n$ is an odd multiple of $m/2$ (including $n = m/2$) then $A$ is inscribed in $B$.*
(b) *If* (i) *$n|m$ or* (ii) *$m > n$ and $m$ is an odd multiple of $n/2$ then there is at least one, and at most two, point(s) of contact on every edge of $B$.*

*Proof.*

(a) If $m|n$, we have seen in the proof of Proposition 4.4 that all the vertices of $A$ are also vertices of $B$. If $n = m/2$ then $n|m$ so every edge of $B$ contains an edge of $A$, but this implies that $A$ is inscribed in $B$ since $m = 2n$. If $n > m$ and $n$ is an odd multiple of $m/2$ then $k = \gcd(m, n) = m/2$ and so the number of points of contact is $2k$, which is $m$. Since in this case $A$ and $B$ do not share a vertex, it follows that every vertex of $A$ is in contact with an interior point of an edge of $B$.
(b) If $n|m$, we have seen in the proof of Proposition 4.4 that every edge of $B$ contains an edge of $A$, so there are two vertices of $A$ on every edge of $B$. If $m > n$ and $m$ is an odd multiple of $n/2$ then $k = \gcd(m, n) = n/2$ and so the number of points of contact is $2k$, which is $n$. So $n$ vertices of $A$ are in contact with $B$. Since in this case $A$ and $B$ do not share an edge or a vertex, it follows that each edge of $B$ contains exactly one vertex of $A$.

□

**Corollary 4.9.** *Set $m = pn + q$, where $0 \le q < n$ and $p = p_*, p_* + 1, \ldots$, where $p_*$ is the smallest integer such that $p_* n + q \ge 3$. Then for fixed $n$ and $q$ and $p = p_*, p_* + 1, \ldots$ we have the following:*

(a) *The sequence $\{\rho_{pn+q,n}\}$ is monotone decreasing with limit* $\cos(\pi/n)$ (see Eq. (2.2)).
(b) *For $q = 0$ (so $n|m$), the ratio of areas $\{R_{pn+q,n}\}$ is a monotone decreasing sequence, with limit $R_{\infty n}$* (see Eq. (2.2)).
(c) *For $q > 0$, the sequence $\{R_{pn+q,n}\}$ is monotone increasing with limit $R_{\infty n}$.*

*Proof.* We note that $k = \gcd(q, n)$ and is fixed as $p$ varies. The proof of (a) is immediate, by inspection of Eq. (3.1) as $m$ increases for fixed $n$ and $k$. To prove (b), note that $k = n$ so

$$R_{mn} = \frac{m}{n} \frac{\sin(2\pi/m)}{\sin(2\pi/n)} \frac{\cos^2 \pi/n}{\cos^2 \pi/m} = \frac{m}{n} \frac{\tan(\pi/m)}{\tan(\pi/n)}. \tag{4.2}$$

Define $\alpha = \pi/m$, so $\alpha \in (0, \pi/3]$ for $m \geq 3$ and $R_{mn} \propto R' \equiv \tan\alpha/\alpha$. Then

$$\frac{dR'}{d\alpha} = \frac{\alpha \sec^2 \alpha - \tan \alpha}{\alpha^2} = \frac{2\alpha - \sin(2\alpha)}{2\alpha^2 \cos^2 \alpha} \geq 0. \tag{4.3}$$

Hence $R_{mn}$ is a decreasing function of $\alpha$ as $\alpha \downarrow 0$. To prove (c), we employ a similar argument. We define $\beta = k/n$. Then

$$R_{mn} = \frac{m}{n} \frac{\sin(2\pi/m)}{\sin(2\pi/n)} \frac{\cos^2(\pi/n)}{\cos^2(\pi\beta/m)}. \tag{4.4}$$

Then $R_{mn} \propto R'' \equiv \sin(2\alpha)/\alpha \cos^2(\alpha\beta)$. We now establish that $dR''/d\alpha < 0$, noting that $\beta$ is fixed, $\beta \in (0, \frac{1}{2}], \sin(\alpha\beta) < \sin\alpha$ and $\cos(\alpha\beta) > \cos\alpha$. Then

$$\begin{aligned}
\frac{dR''}{d\alpha} &= \frac{2\cos(2\alpha)}{\alpha\cos^2(\alpha\beta)} - \frac{\sin(2\alpha)}{\alpha^2\cos^2(\alpha\beta)} + \frac{2\beta\sin(2\alpha)\sin(\alpha\beta)}{\alpha\cos^3(\alpha\beta)} \\
&< \frac{2\cos(2\alpha)}{\alpha\cos^2(\alpha\beta)} - \frac{\sin(2\alpha)}{\alpha^2\cos^2(\alpha\beta)} + \frac{\sin(2\alpha)\sin\alpha}{\alpha\cos\alpha\cos^2(\alpha\beta)} \\
&= \frac{2(\alpha - \tan\alpha)\cos^2\alpha}{\alpha^2\cos^2(\alpha\beta)} \\
&< 0.
\end{aligned} \tag{4.5}$$

Hence $R_{mn}$ is a increasing function of $\alpha$ as $\alpha \downarrow 0$. In all cases, the limits as $m \to \infty$ are obvious. □

*Remark* 4.10. It may be of interest to observe that Corollary 4.9 implies the following consequence for dense packings of identical regular polygons in the plane. In the special case $n = 6$ and $m = 6p$ ($p = 1, 2, \ldots$), a packing of the plane using identical regular $(6p)$-gons, arranged in a hexagonal tiling, has a *higher* packing density than that of identical circles. The special case of 12-gons is known as the 'truncated hexagonal' tiling. The proof is immediate, since for $p = 1, 2, \ldots$, the area ratio of the $(6p)$-gons is a monotone *decreasing* sequence with limit equal to that of the inscribed circle of a hexagon.

## 5. Triangle in polygon ($m = 3$)

The following lemma is the key to our results in this section. (It is an application of van Schooten's theorem [4].)

**Lemma 5.1.** *Let $T_1$ and $T_2$ be triangles with $T_2 \subset T_1$. Suppose that $T_1$ and $T_2$ do not share a vertex. Then $T_2$ may be moved (by an arbitrarily small displacement) into the interior of $T_1$.*

*Proof.* If one vertex of $T_2$ is contained in the interior of $T_1$, then clearly we may translate $T_2$ slightly so that it is entirely contained in the interior of $T_1$. If $T_1$ and $T_2$ share an edge then we may translate $T_2$ slightly, parallel to the shared edge, so that the third vertex moves into the interior of $T_1$. So we may assume that all three vertices of $T_2$ lie on the interiors of the three edges of $T_1$.

Suppose that two vertices of $T_2, \mathcal{A}$ and $\mathcal{B}$ say, may slide slightly in both directions along sides of $T_1$ which emanate from a vertex $\mathcal{C}'$ of $T_1$. Van Schooten's theorem [4] asserts that as $\mathcal{A}$ and $\mathcal{B}$ slide freely the third vertex $\mathcal{C}$ of $T_2$ describes a portion of an ellipse whose centre is the vertex $\mathcal{C}'$ of $T_1$. In particular, if the third side $\mathcal{A}'\mathcal{B}'$ of $T_1$ is tangential to this ellipse at $\mathcal{C}$, then the ellipse and $T_1$ will be on the same side of $\mathcal{A}'\mathcal{B}'$. So $\mathcal{C}$ will move into the interior of $T_1$ as $\mathcal{A}$ and $\mathcal{B}$ slide slightly away (in either direction) from their initial positions. On the other hand, if $\mathcal{A}'\mathcal{B}'$ is transverse to the ellipse at $\mathcal{C}$, then we may slide $\mathcal{A}$ and $\mathcal{B}$ in *one* direction so that $\mathcal{C}$ moves into the interior of $T_1$.

Finally, suppose that no two vertices of $T_2$ may slide freely in both directions along the corresponding sides of $T_1$. This implies that each side of $T_2$ is perpendicular to a corresponding side of $T_1$. But then we may rotate $T_2$ slightly about any one of its vertices so that the other two vertices move into the interior of $T_1$. □

**Proposition 5.2.** *Suppose that $T$ is a triangle, $P$ is a convex polygon, and $T \subseteq P$. If $T$ and $P$ do not share a vertex or part of an edge then $T$ may be moved (by an arbitrarily small displacement) into the interior of $P$.*

*Proof.* We may suppose as before that the vertices of $T$ are interior points of edges of $P$. Suppose that the three edges of $P$ containing the vertices of $T$ may be extended to form a triangle (necessarily containing $T$). By Lemma 5.1 we may move $T$ into the interior of the larger triangle by an arbitrarily small displacement. In particular, fixed small neighbourhoods of the vertices of $T$ are moved into the interior of $P$ whenever the displacement is sufficiently small. This implies, provided the displacement is sufficiently small, that the whole of $T$ will be moved into the interior of $P$. On the other hand, if two of the edges diverge away from the third edge or are parallel, then we may translate $T$ slightly, parallel to one or other of the diverging edges and away from the third edge, so that the vertex of $T$ on the third edge moves into the interior of $P$. □

The next corollary will be used in the case $m = 4$ and in the case of coprime $m$ and $n$ considered below.

**Corollary 5.3.** *Suppose that $P_1$ and $P_2$ are convex polygons with $P_2 \subseteq P_1$. If $P_1$ and $P_2$ do not share a vertex or part of an edge and at most three vertices of $P_2$ are on edges of $P_1$, then $P_2$ may be moved (by an arbitrarily small displacement) into the interior of $P_1$.*

*Proof.* Let $T$ be a triangle formed by the vertices of $P_2$ which are on edges of $P_1$ (if any) and any other vertices of $P_2$ (if needed). By Proposition 5.2 we may displace $T$ slightly so that $T$ moves into the interior of $P_1$. Provided this displacement is sufficiently small it follows that the displaced copy of $P_2$ will be contained in the interior of $P_1$. □

**Corollary 5.4.** *Suppose that* $3 = m < n$. *Then* $A$ *and* $B$ *share a vertex and a common axis of symmetry through that vertex.*

*Proof.* Clearly $A$ and $B$ cannot share part of an edge. By optimality, it is not possible to move $A$ into the interior of $B$. Hence by Proposition 5.2 $A$ and $B$ share a vertex. Since $A$ is equilateral and all its vertices lie on edges of $B$ it follows that $A$ and $B$ have a common axis of symmetry through the shared vertex. □

The radius of $A$ is straightforward to determine. We omit the details. We may assume that $n$ is of the form $3j \pm 1$ as the case where $m$ is divisible by 3 is covered by Theorem 4.5. In terms of the notation of Eq. (2.1), the radius for $n = 3j \pm 1$ is given by

$$\rho_{3n} = \frac{\cos(\pi/n) + \cos(\pi/3n \mp \pi/3)}{\cos(\pi/3n) + \cos(\pi/3n \mp \pi/3)}. \tag{5.1}$$

Since the polygons share a vertex, the optimal configurations are all "radius to radius." If $n$ is odd, the optimal configurations are also "apothem to apothem."

## 6. Polygon in triangle ($n = 3$)

**Corollary 6.1.** *Suppose that* $3 = n < m$. *Then* $A$ *and* $B$ *share an edge and share the axis of symmetry perpendicular to that edge.*

*Proof.* To derive a contradiction, suppose that $A$ and $B$ do not share an edge. Clearly, $A$ and $B$ cannot share a vertex. But at most three vertices of $A$ lie on edges of $B$ (since $B$ is a triangle). Hence by Corollary 5.3 $A$ may be moved into the interior of $B$, contradicting optimality. Consider the axis of symmetry of $A$ through the midpoint of the shared edge. If that axis is translated perpendicularly to the shared edge until it coincides with the parallel axis of symmetry of $B$, then the half of $A$ which moves towards the axis of symmetry of $B$ will (by convexity) remain contained in $A$. Hence, by symmetry, the other half of $A$ will also be contained in $B$ after the translation. Now $A$ and $B$ share an axis of symmetry. Clearly, this is the unique optimal configuration sharing that edge. □

The radius of $A$ is also straightforward to determine, and we again omit the details. Clearly $m$ must be of the form $3j \pm 1$. In terms of the notation of Eq. (2.1), the answer is, setting $j_3 = [m/3]$ (the largest integer less than or equal to $m/3$)

$$\rho_{m3} = \frac{3/2}{\cos(\pi/m) - 2\cos(\pi/m + 2\pi j_3/m + \pi/3)}. \tag{6.1}$$

Curiously, this formula also works if $m$ is divisible by 3, i.e., it is applicable for *all* $m \geq 3$, for fixed $n = 3$. Since the polygons share an edge, the optimal configurations are all "apothem to apothem." If $m$ is odd, the optimal configurations are also "radius to radius."

## 7. Square $A$ inside regular polygon $B(m = 4, n > 4)$

In this section we find the largest square $A$ (i.e., $m = 4$) contained in a regular polygon $B$ with $n > 4$ sides. (The case $n = 3$ was considered above.) The configurations considered will be shared-vertex, radius–apothem and apothem–apothem. We assume $n$ is odd, for otherwise $m$ and $n$ are both even, and the solution is given by Theorem 4.5. Use Fig. 1 as a reference. (The square with a shared vertex is optimal.) It shows shared-vertex and apothem–apothem configurations (it is easy to visualize the radius–apothem case). The solution for $(m, n) = (4, 5)$ (square in pentagon) is shown in Fig. 2a. We shall show that $(m, n) = (4, 5)$ and $(m, n) = (4, 9)$ are the only two cases where the optimal solution has a shared vertex and only three points of contact.

By Corollary 5.3 if $A$ and $B$ do not share a vertex then *every* vertex of the square $A$ is a point of contact with $B$. It will first be shown that $A$ and $B$ share a common axis of symmetry (which we henceforth take to be the $x$-axis). Then we know that the optimal $A$ will be one of these three candidates: shared-vertex, radius–apothem, or apothem–apothem. Our goal is then to find expressions for the radius of $A$ in the three configurations and to determine which is the largest.

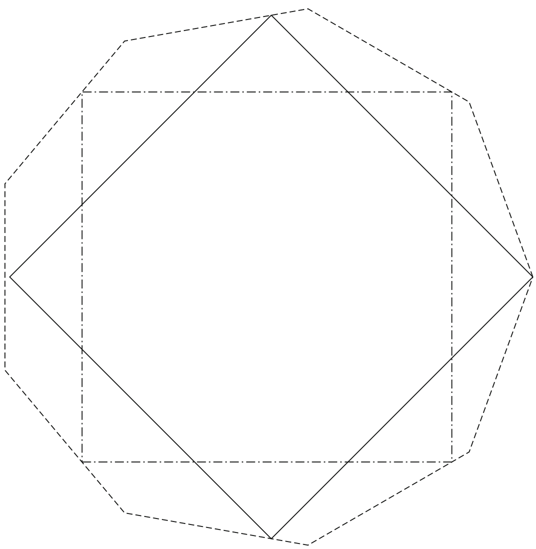

FIGURE 1 A square in a nonagon, i.e., $(m, n) = (4, 9)$, showing candidates for the shared-vertex and apothem–apothem configurations. The square with a shared vertex is optimal

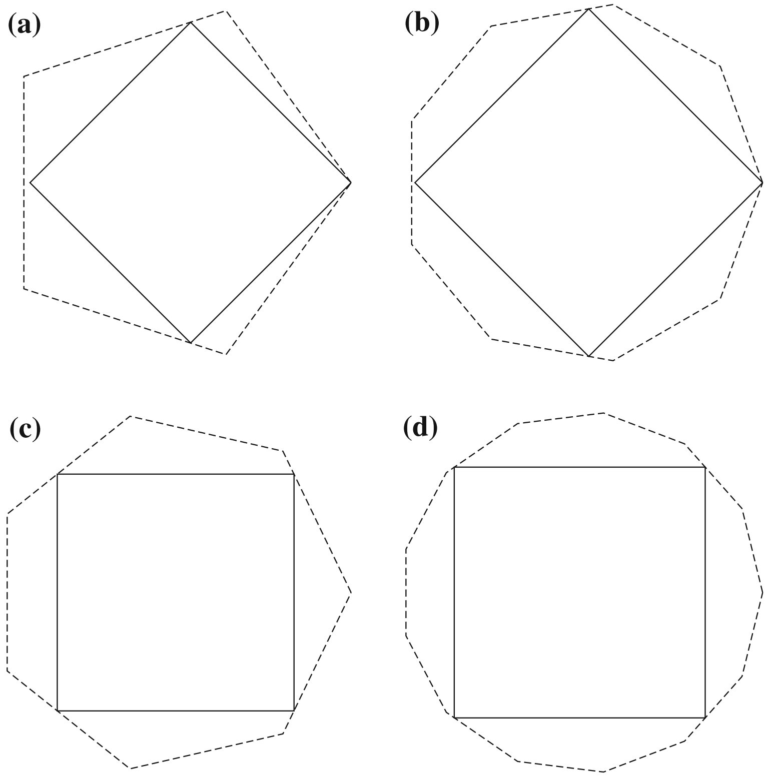


FIGURE 2 Examples of square in polygon, i.e., $m = 4$, for nonconcentric polygons, i.e., odd $n$. Here $n = 5, 9, 7, 13$ in **a**, **b**, **c** and **d** respectively, where **a** and **b** demonstrate the two exceptional cases where the polygons share a common vertex (and the polygons have three points of contact). The cases **c** and **d** are representative of all the other cases of odd $n$, i.e., apothem–apothem configuration and four points of contact (hence the square is inscribed in the outer polygon)

### 7.1. Proof of existence of common axis of symmetry

**Definition 7.1.** Let $L_1$ and $L_2$ be two straight lines in the complex plane which pass through the origin and which are not perpendicular to each other. For $z \in \mathbb{C}$ we define a non-negative function $F(z)$ as follows. The positively (resp. negatively) oriented *perpendicular distance* $\sqrt{F(z)}$ from $z$ to $L_1$ and $L_2$ is defined to be the common length of two perpendicular line segments of equal length emanating from $z$ and terminating on $L_1$ and $L_2$ in points $x$ and $y$ such that the triangle with vertices $(x, y, z)$ taken in that order is positively (resp. negatively) oriented.

**Lemma 7.2.** *If $w_0 \neq 0$ and $w_1 \neq 0$ then $F(w_0 + tw_1)$ is a positive quadratic function of $t \in \mathbb{R}$.*

*Proof.* Assume without loss of generality that $\sqrt{F(z)}$ is the positively oriented perpendicular distance. Suppose $L_1 = \{xa : x \in \mathbb{R}\}$ and $L_2 = \{yb : y \in \mathbb{R}\}$. For $z \in \mathbb{C}$, let $x(z)a$ and $y(z)b$ be the endpoints on $L_1$ and $L_2$ of the perpendicular line segments of equal length emanating from $z$. Then

$$z = \frac{x(z)a + y(z)b}{2} + i\left(\frac{y(z)b - x(z)a}{2}\right) \tag{7.1}$$

and

$$F(z) = |z - x(z)a|^2.$$

Provided $b/a$ is not purely imaginary, i.e., provided $L_1$ and $L_2$ are not perpendicular to each other, Eq. (7.1) has a unique solution $(x(z), y(z))$ for each $z \in \mathbb{C}$. Setting $z = w_0 + tw_1$ in Eq. (7.1) and taking real and imaginary parts we find that $x(z) = mt + c$ for some $m, c \in \mathbb{R}$. Hence

$$F(w_0 + tw_1) = |(w_0 - ca) + t(w_1 - ma)|^2,$$

which is a positive quadratic function of $t$. □

**Proposition 7.3.** *Suppose that $A$ is the largest square contained in $B$. Then $A$ and $B$ share an axis of symmetry.*

*Proof.* We may assume $n$ is odd. If $A$ and $B$ share a vertex, it is clear that there is a common axis of symmetry through that vertex. So we may assume that $A$ and $B$ do not share a vertex. It follows from Corollary 5.3 that every corner of $A$ lies on the interior of an edge of $B$.

Let us identify any fixed edge of $B$ with the unit interval $[0, 1]$. As $x$ increases from 0 to 1 the corresponding point (also denoted $x$) on the edge of $B$ moves counterclockwise from one vertex to an adjacent vertex. From each point $y$ on $B$ there is a unique pair of perpendicular chords of equal length emanating from $y$ whose endpoints $p(y)$ and $q(y)$ lie on edges of $B$. Let $\sqrt{F(y)}$ be the common length of these chords. Restricting attention to the edge $[0, 1]$, there exist $0 < x_0 < x_1 < 1$, symmetrically placed about $1/2$, such that $p(x_0)$ and $q(x_1)$ are vertices of $B$. On each of the three subintervals $[0, x_0]$, $[x_0, x_1]$, and $[x_1, 1]$, we see that $\sqrt{F(x)}$ is an oriented 'perpendicular distance' from the point $x$ to the pair of lines formed by extending the edges containing $p(x)$ and $q(x)$. By Lemma 7.2 the restriction of $F(x)$ to each subinterval is a positive quadratic function. It follows that in each subinterval $F(x)$ has at most one stationary value which is necessarily a local minimum. This implies that the nonempty level sets of $F(x)$ $(0 \le x \le 1)$ consist of $k$ 'pairs' $(1 \le k \le 3)$ of points symmetrically placed about $1/2$. (Recall that a *level set* of a function is the set of points at which the function assumes a prescribed value.) Let $x^*$ denote the other member of the pair containing $x$, i.e., the reflection of $x$ in $1/2$. (Note that $x^* = x$ if $x = 1/2$.)

Now consider the optimal square $A$. Note that $F(y)$ takes the same value at all four corners of $A$. The level set $S$ corresponding to this common value of $F(y)$ consists of $k$ pairs, where $1 \le k \le 3$. First suppose $k = 3$, so that $S = \{a, a^*, b, b^*, c, c^*\}$. We shall say that two points $y_1, y_2$ on $B$ are *of the* *same type* if $y_2$ is a rotation of $y_1$ through an integer multiple of $2\pi/n$, i.e.,, $y_1$ and $y_2$ are in the same relative positions on their respective edges.

Suppose that two adjacent corners $x$ and $y$ of $A$ are of the same type, $a$, say. Clearly, $p(x)$ and $p(y)$ are of the same type as each other, and $q(x)$ and $q(y)$ are also of the same type as each other, and all are corners of $A$. But $p(x)$ is of type $a$ and $q(y)$ is also of type $a$ since $x$ and $y$ are adjacent corners of type $a$. So all four corners of $A$ are of the same type. But this forces $n$ to be divisible by 4 since as we move around $B$ from one corner of $A$ to an adjacent corner of $A$ we always pass through the same number of vertices of $B$. This contradicts our assumption that $n$ is odd.

Next suppose that two diagonally opposite corners $x$ and $y$ of $A$ are of the same type $a$, say. Since $p(x)$ and $p(y)$ will also be of the same type, this forces the other two corners of $A$ to be of the same type, $b$ say. So moving counter-clockwise around $A$ we pass through vertices of type $abab$ in that order. But this forces $n$ to be even, since we move from an $a$ type to a $b$ type twice and from a $b$ type to an $a$ type twice. This again contradicts our assumption that $n$ is odd.

So we have shown that no two corners of $A$ are of the same type. Then since $S$ has six elements, it follows from the pigeon-hole principle that the types of two corners of $A$ must from a pair $\{a, a^*\}$, say. The perpendicular bisector of the line joining these two corners will be a common axis of symmetry for $A$ and $B$.

The proofs in the cases $k = 1$ and $k = 2$ are easier. □

**7.2. Shared-vertex**

In all the calculations below, we place the origin at the centre of $B$ and one vertex of $B$ at the coordinate location $(1, 0)$. To calculate the radius of $A$, we begin with the shared-vertex configuration. Label the vertices of $B$ as $j = 0, 1, 2, \ldots$ counting counterclockwise from $j = 0$ on the $x$-axis. Let the centre of $A$ be at $(x_{sv}, y_{sv})$. We know $y_{sv} = 0$. The vertex at $(1, 0)$ is a shared vertex (by definition). The vertex in the upper half-plane $y > 0$, is given by the intersection of the straight lines $x + y = 1$ and (for some $j > 0$)

$$\frac{y - \sin(2j\pi/n)}{x - \cos(2j\pi/n)} = -\frac{1}{\tan((2j+1)\pi/n)}.$$

Let the point of intersection be $(x_*, y_*)$, and suppose it occurs for $j = j_*$. The solution for $y_*$ is also the radius $\rho_{sv}$ of $A$. It is easily seen that $j_* = [n/4]$, i.e., the largest integer less than (or equal to) $n/4$, but equality never occurs because $n$ must be odd. If $n = 4j_* + 1$ the radius is

$$\rho_{sv} = \frac{\cos(\pi/n) + \sin(\pi/2n)}{\cos(\pi/2n) + \sin(\pi/2n)]}, \tag{7.2}$$

whereas if $n = 4j_* + 3$, the radius is

$$\rho_{sv} = \frac{\cos(\pi/n) - \sin(\pi/2n)}{\cos(\pi/2n) - \sin(\pi/2n)}.$$

Note that the formula for $n = 4j_* + 3$ is obtained from the formula for $n = 4j_* + 1$ by reversing the sign of $n$ (or $\sin(\pi/2n)$).

### 7.3. Radius–apothem

We include this case just for completeness, although it is never optimal. For the radius–apothem case (vertex of $A$ touches midpoint of edge of $B$), let the point of contact be on the negative $x$-axis, at $x = -\cos(\pi/n)$. The other points of contact are at edges of $B$ indexed by $j_{**} = [(n+2)/4]$. If $n = 4j_{**} + 1$, the radius is

$$\rho_{ra} = [1 - \sin(\pi/2n)][\cos(\pi/2n) + \sin(\pi/2n)].$$

If $n = 4j_{**} - 1$, the radius is

$$\rho_{ra} = [1 + \sin(\pi/2n)][\cos(\pi/2n) - \sin(\pi/2n)].$$

We again simply reverse the sign of $n$ (or $\sin(\pi/2n)$).

### 7.4. Apothem–apothem

The final case is apothem–apothem. There are four points of contact. One point of contact occurs at an edge of $B$ indexed by $j_1 = [n/8]$. For $n = 4j_1 + 1 = 5, 9, 13, 17, 21, \ldots$, the radius is

$$\rho_{aa} = \frac{[1 - \sin(\pi/2n)][\cos(\pi/2n) + \sin(\pi/2n)]}{\cos(\pi/4n)}. \tag{7.3}$$

For $n = 4j_1 - 1 = 3, 7, 11, 15, 19, \ldots$, the radius is

$$\rho_{aa} = \frac{[1 + \sin(\pi/2n)][\cos(\pi/2n) - \sin(\pi/2n)]}{\cos(\pi/4n)}. \tag{7.4}$$

As before, we reverse the sign of $n$ (or $\sin(\pi/2n)$). We see that for all odd $n$,

$$\frac{\rho_{aa}}{\rho_{ra}} = \frac{1}{\cos(\pi/4n)} > 1.$$

Hence radius–apothem is never optimal. For $n = 4j_1 - 1$ we furthermore have that

$$\begin{aligned}\frac{\rho_{ra}}{\rho_{sv}} &= \frac{[1 + \sin(\pi/2n)][\cos(\pi/2n) - \sin(\pi/2n)]^2}{\cos(\pi/n) - \sin(\pi/2n)} \\ &= \frac{[1 + \sin(\pi/2n)][1 - 2\cos(\pi/2n)\sin(\pi/2n)]}{[1 + \sin(\pi/2n)][1 - 2\sin(\pi/2n)]} \\ &= \frac{1 - 2\cos(\pi/2n)\sin(\pi/2n)}{1 - 2\sin(\pi/2n)} \\ &> 1.\end{aligned}$$

Hence $\rho_{aa} > \rho_{ra} > \rho_{sv}$ and so apothem–apothem is always the optimal configuration for $n = 4j - 1$.

Next we compare apothem–apothem to the shared-vertex case for $n = 4j + 1$. Recall that in this case (see Eq. (7.2))

$$\rho_{sv} = \frac{\cos(\pi/n) + \sin(\pi/2n)}{\cos(\pi/2n) + \sin(\pi/2n)}.$$

Then

$$\begin{aligned} \frac{\rho_{aa}}{\rho_{sv}} &= \frac{1}{\cos(\pi/4n)} \frac{[1 - \sin(\pi/2n)][\cos(\pi/2n) + \sin(\pi/2n)]^2}{1 - 2\sin^2(\pi/2n) + \sin(\pi/2n)} \\ &= \frac{1}{\cos(\pi/4n)} \frac{[1 - \sin(\pi/2n)][1 + 2\sin(\pi/2n)\cos(\pi/2n)]}{[1 - \sin(\pi/2n)][1 + 2\sin(\pi/2n)]} \\ &= \frac{1}{\cos(\pi/4n)} \frac{1 + 2\sin(\pi/2n)\cos(\pi/2n)}{1 + 2\sin(\pi/2n)}. \end{aligned}$$

Setting $\xi = \pi/2n$, we obtain

$$\frac{\rho_{aa}}{\rho_{sv}} = \frac{1}{\cos(\xi/2)} \frac{1 + 2\sin\xi\cos\xi}{1 + 2\sin\xi} = \frac{g(\xi)}{f(\xi)},$$

where we define

$$\begin{aligned} f(\xi) &= \cos(\xi/2), \\ g(\xi) &= \frac{1 + 2\sin\xi\cos\xi}{1 + 2\sin\xi}. \end{aligned}$$

Then $f = g = 1$ for $\xi = 0$. Also $f$ and $g$ take values in $[0, 1]$ and are decreasing functions of $\xi$ for $\xi \in [0, \pi/2]$. In fact we need only consider $n \geq 3$ so $\xi \leq \pi/6$. It is a matter of algebra to establish that $\rho_{aa}/\rho_{sv} > 1$ for $n = 4j + 1 \geq 13$, but $\rho_{aa}/\rho_{sv} < 1$ for $n = 5$ and 9. The graphs of $f(\xi)$ (solid) and $g(\xi)$ (dashed) are plotted against $2\xi/\pi$ (i.e., $1/n$) for $0 \leq 1/n \leq 0.25$, in Fig. 3. There is only one crossing point. The solutions for $n = 5$ and 9 are to the right of the crossing point, as indicated in the figure; all other values of $n = 4j + 1$ are to the left, i.e., $f < g$ or $\rho_{aa} > \rho_{sv}$. To summarize, the optimal configurations are "radius to radius" (with a shared vertex) for $n = 5$ and 9, and are "apothem to apothem" for all other odd $n \geq 3$ (and for $n = 3$ only, the polygons share an edge).

## 8. Regular polygon $A$ in square $B$ ($n = 4$)

The case of $n = 4$, a regular polygon in a square, was solved by Dureisseix [3]. He showed that the optimal polygon $A$ shares an axis of symmetry with a diagonal of the square $B$ (for any $m$). The solution for even $m$ is a special case of our general solution for $\gcd(m, n) > 1$. If $m$ is odd then $m$ and $n$ are coprime. In terms of the notation of Eq. (2.1), the optimal radius of $A$ is (see

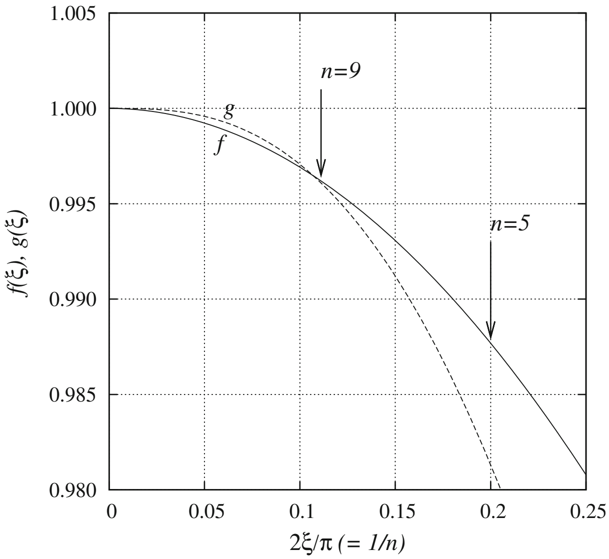


FIGURE 3 Graphical solution to aid in the determination of the optimum for the shared-vertex and apothem–apothem configurations for a square in a polygon, for nonconcentric polygons. Here $\xi = \pi/2n$ and the functions $f(\xi)$ (*solid*) and $g(\xi)$ (*dashed*) are defined in the text

also the top row of Table 1 in [3])

$$\rho_{m4} = \frac{1}{\sqrt{2}\cos(\pi/2m)\cos(\pi/4m)}. \tag{8.1}$$

All of the optimal configurations are "radius to radius." In the case $m = 3$ (triangle in square) the polygons share a common vertex, but for $m \geq 5$ the polygons do not share any common vertices.

## 9. General solution for coprime $m$ and $n$

We treat the general case of coprime $m$ and $n$. Now the optimal polygons are never concentric. It is convenient to define $\Delta := \pi/mn$ for later use below. Then there always exists a pair of symmetry axes, of $A$ and $B$ respectively, such that the angle between them, say $\theta$, lies in the interval $\theta \in [0, \Delta]$. We now state a crucial assumption, which we have unfortunately not succeeded in proving.

**Assumption 9.1.** *For a fixed relative orientation $\theta \in (0, \Delta)$ of the polygons, the largest regular $m$-gon contained in the polygon $B$ has at most three points of contact with $B$, all of which lie on interior points of edges of $B$*

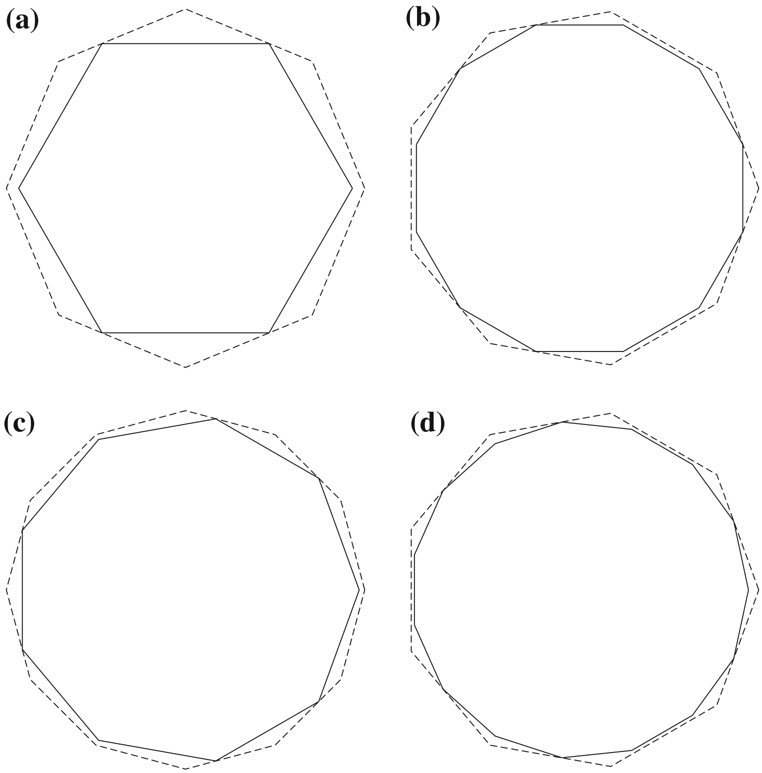


FIGURE 4 Examples of concentric polygons demonstrating radius–radius and apothem–apothem configurations, **a** $(m,n) = (6,8)$: $k = \gcd(m,n) = 2, m/k =$ odd, $n/k =$ even, radius–radius, **b** $(m,n) = (12,9)$: $k = \gcd(m,n) = 3, m/k =$ even, $n/k =$ odd, apothem–apothem, **c** $(m,n) = (9,12)$: $k = \gcd(m,n) = 3, m/k =$ odd, $n/k =$ even, radius–radius, **d** $(m,n) = (15,9)$: $k = \gcd(m,n) = 3, m/k =$ odd, $n/k =$ odd, simultaneous radius–radius and apothem–apothem

It then follows by Corollary 5.3 that the optimal polygon $A$ cannot have an orientation $\theta \in (0, \Delta)$. The largest regular $m$-gon in $B$ must have an orientation either $\theta = 0$ or $\theta = \Delta$. (In both of the above cases the polygons $A$ and $B$ share a common symmetry axis.) Numerical searches have not revealed any counterexamples to the above assumption, but we have no explicit proof. For the record we state the answer as a conjecture, without proof. All of the results proved above for $m = 3, n = 3, m = 4$ (and $n = 4$ by Dureisseix [3]), as well as Daley's results for $m = n \pm 1$ [2], are special cases of the results below.

The largest $A$ in $B$ always shares a symmetry axis with $B$, and its centre lies on that common symmetry axis. Orient $A$ and $B$ such that a radius of $A$ is aligned with an apothem of $B$. Denote this orientation by $\theta = 0$. There is another possible orientation, where a radius of $A$ makes an angle of $\Delta$ with

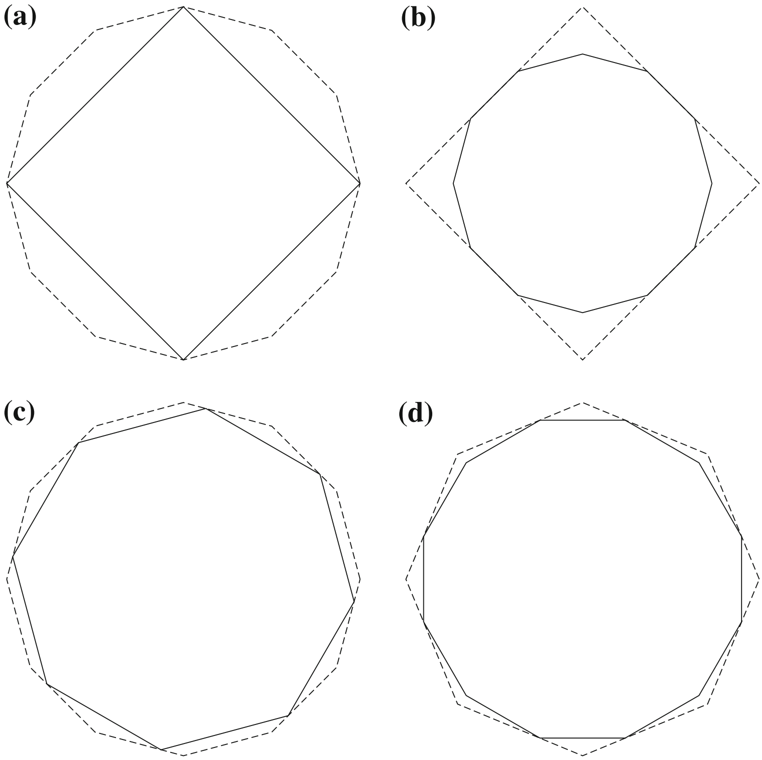


FIGURE 5 Examples of concentric polygons demonstrating the configurations in Corollary 4.8: **a** $(m, n) = (4, 12)$: $m|n$, all vertices of the inner polygon are vertices of the outer polygon, **b** $(m, n) = (12, 4)$: $n|m$, every edge of the outer polygon contains an edge of the inner polygon, **c** $(m, n) = (8, 12)$: $n$ is an odd multiple of $m/2$, no shared edges or vertices, **d** $(m, n) = (12, 8)$: $m > n$ and $m$ is an odd multiple of $n/2$. In **a** and **c** the inner polygon is inscribed in the outer, while in **b** and **d** every edge of the outer polygon contains at least one vertex of the inner polygon

an apothem of $B$ (denote this orientation by $\theta = \Delta$). (In this latter case the common symmetry axes may be radius–radius, apothem–radius or apothem–apothem.) Because $m$ and $n$ are coprime, there exist unique integers $1 \le i_1 < m$ and $1 \le j_1 < n$ such that $j_1 m - i_1 n = 1$. One can show that either (i) both $i_1 < m/2$ and $j_1 < n/2$, or else (ii) both $i_1 > m/2$ and $j_1 > n/2$. Then define integers $1 \le i_* < m/2$ and $1 \le j_* < n/2$ such that

$$(i_*, j_*) = \begin{cases} (i_1, j_1) & \text{if} \quad i_1 < m/2, j_1 < n/2 \\ (m - i_1, n - j_1) & \text{if} \quad i_1 > m/2, j_1 > n/2. \end{cases} \tag{9.1}$$

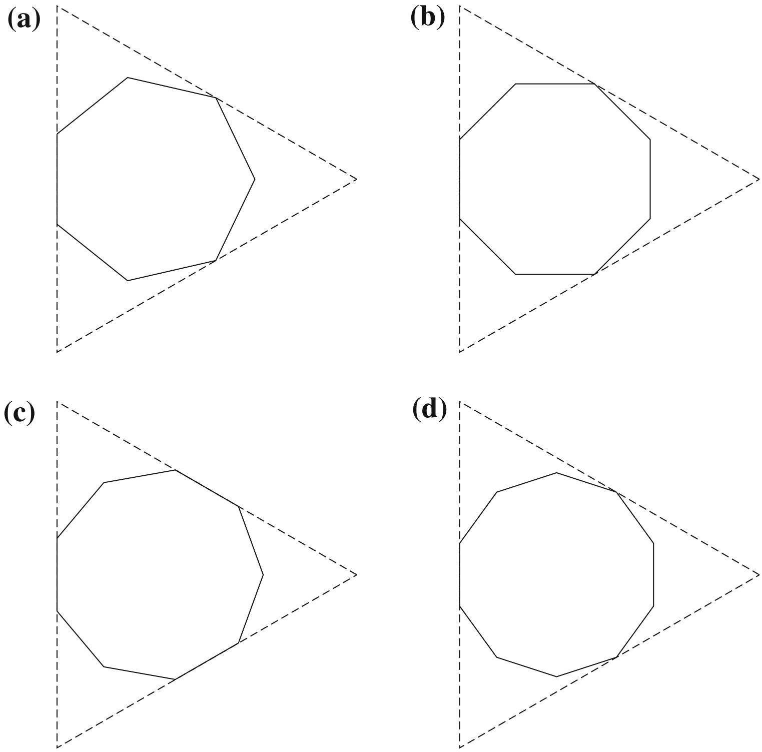


FIGURE 6 Examples of polygon in triangle, i.e., $n = 3$, demonstrating shared edge, where $m = 7, 8, 9, 10$ in **a**–**d** respectively. For $m = 9$ the polygons are concentric (and share three edges), but for $m$ not divisible by 3 the polygons are not concentric and share only one edge. For even $m$ (**b** and **d**) the configurations are apothem–apothem. For odd $m$ (**a** and **c**) the configurations are simultaneously radius–radius and apothem–apothem

Then there exist integers $\mu$ and $\nu$ which simultaneously satisfy all of the following inequalities:

$$\mu \leq \frac{m}{4i_*} \leq \mu + 1, \qquad \mu \leq \frac{n}{4j_*} \leq \mu + 1, \tag{9.2}$$

and

$$\nu - \frac{1}{2} \leq \frac{m}{4i_*} \leq \nu + \frac{1}{2}, \qquad \nu - \frac{1}{2} \leq \frac{n}{4j_*} \leq \nu + \frac{1}{2}. \tag{9.3}$$

Solutions for $\mu$ and $\nu$ always exist. (One can also show that either $\nu = \mu$ or $\mu+1$.) Since $i_* < m/2$ and $j_* < n/2$, we must have $\nu \geq 1$. However if $i_* > m/4$ or $j_* > n/4$ (which can happen), then we have $\mu = 0$. If $\mu = 0$, then the optimal orientation is always $\theta = \Delta$ and the radius $\rho_{\max}$ of the largest polygon $A$ contained in $B$ is

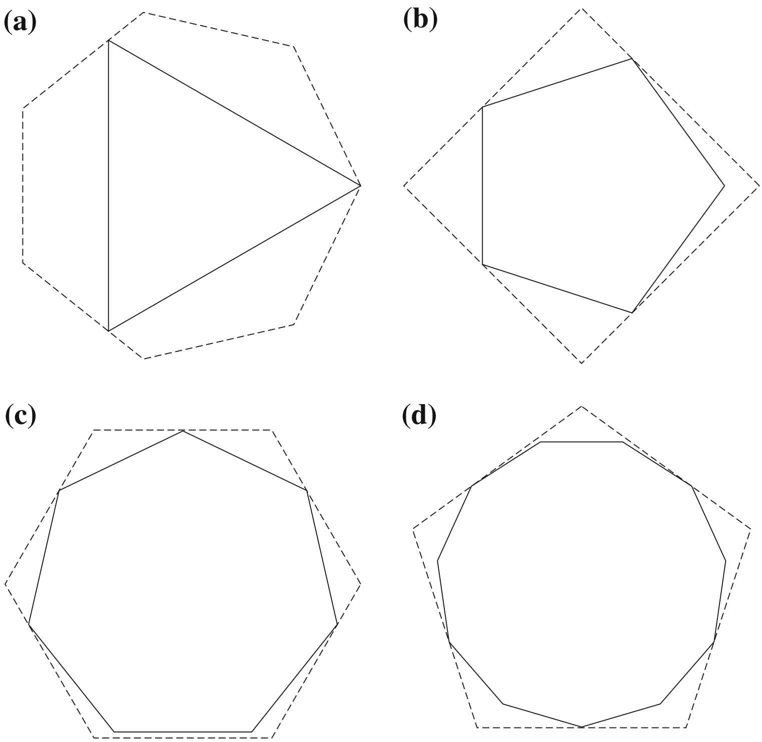


FIGURE 7 Examples of nonconcentric polygons demonstrating radius–radius and apothem–apothem configurations, and also an example which is neither radius–radius nor apothem–apothem. (Similar in concept to Fig. 4, but for nonconcentric polygons.) **a** $(m, n) = (3, 7)$: simultaneous radius–radius and apothem–apothem (also nonconcentric inscribed polygon with shared vertex), **b** $(m, n) = (5, 4)$: radius–radius, **c** $(m, n) = (7, 6)$: apothem–apothem, **d** $(m, n) = (11, 5)$: neither radius–radius nor apothem–apothem. The common bisector is the $x$ axis in **a** and **b** and is the $y$ axis in **c** and **d**. Note that in **c** there are *four* points of contact; the *top vertex* of the heptagon is *not* in contact with the top edge of the hexagon. Also in **d** there are four contacts; the *bottom vertex* of the inner polygon is not in contact with the bottom edge of the outer polygon. Note that Figs. 2 and 6 and also display examples of nonconcentric polygons, possibly with a shared edge or vertex

$$\rho_{mn} = \frac{\cos(\pi/n)}{\cos\Delta}\,\frac{1-\cos(2j_*\pi/n)}{\cos(2\Delta)-\cos(2j_*\pi/n)}. \tag{9.4}$$

If $\mu > 0$ the solution is more complicated. Define

$$\rho_0 = \frac{2\cos(\pi/n)\sin(j_*\pi/n)\sin((2\mu+1)j_*\pi/n)}{\cos(2(\mu+1)\Delta)\cos(2\mu j_*\pi/n) - \cos(2\mu\Delta)\cos(2(\mu+1)j_*\pi/n)}, \tag{9.5}$$

$$\rho_\Delta = \frac{2\cos(\pi/n)\sin(j_*\pi/n)\sin(2\nu j_*\pi/n)}{\cos((2\nu+1)\Delta)\cos((2\nu-1)j_*\pi/n) - \cos((2\nu-1)\Delta)\cos((2\nu+1)j_*\pi/n)}. \tag{9.6}$$

The radius $\rho_{mn}$ of the largest polygon $A$ contained in $B$ is the larger of these two values:

$$\rho_{mn} = \max\{\,\rho_0, \rho_\Delta\,\}. \tag{9.7}$$

If it is the former, then the orientation is $\theta = 0$, if the latter then it is $\theta = \Delta$. In the statements below, $B$ is fixed to have unit radius (as always), and the "optimal polygon" refers to $A$.

- The optimal polygon $A$ is unique, up to trivial rotations and reflections. If $\mu = 0$, the orientation of the optimal polygon $A$ is always $\theta = \Delta$. If $\mu > 0$, we have not found a simple rule to determine the orientation of the optimal polygon $A$.
- For $m = 3$ (triangle in polygon), the polygons always share a common vertex. For $m = 4$ (square in polygon), the polygons share a common vertex for $n = 5$ and $n = 9$. There are no other cases (for coprime $m$ and $n$) with a shared vertex.
- For $n = 3$ (polygon in triangle), the polygons always share a common edge. There are no other cases (for coprime $m$ and $n$) with a shared edge.
- Except for cases of shared vertices or edges, the polygons always have four points of contact, consisting of vertices of $A$ touching interior points of edges of $B$. If we define the common symmetry axis as the $x$-axis (with the origin at the centre of $B$) there is exactly one contact point in each of the four quadrants.
- Recall that for $\gcd(m,n) > 1$ the optimal configurations are always "apothem–apothem" or "radius–radius" or both. This is *not* always the case for coprime $m$ and $n$; optimal solutions do exist where no radius (resp. apothem) of $A$ is aligned with a radius (resp. apothem) of $B$. This happens, for example, for $m = n \pm 6$. See Fig. 7d, for an example with $(m,n) = (11,5)$.
- Let us write $m = pn+q$ where $0 < q < n$ and $p \geq 0$. Fix $n$ and $q$, then the values of $j_*, \mu$ and $\nu$ are independent of $p$. Define $p_*$ to be the smallest integer such that $p_*n + q \geq 3$. Then for $p = p_*, p_*+1, \ldots$, the sequence $\{\,\rho_{pn+q,n}\,\}$ is strictly decreasing with limit $\cos(\pi/n)$, which is the radius of the inscribed circle of $B$. If the orientation $\theta = 0$ is optimal for $p = p_*$, it remains so for all $p > p_*$. Also if the orientation $\theta = \Delta$ is optimal for $p = p_*$, it remains so for all $p > p_*$. For the ratio of areas, numerical calculations indicate that in all cases the ratio increases monotonically as $m$ increases, for fixed $n$ and $q$.

## 10. Graphs of illustrative examples

Several figures are presented in the text, to illustrate the various cases we have studied in this paper.

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S. J. Dilworth
Department of Mathematics
University of South Carolina
Columbia, SC 29208
USA
e-mail: dilworth@math.sc.edu

S. R. Mane
Convergent Computing Inc.,
P.O. Box 561, Shoreham, NY 11786, USA
e-mail: srmane001@gmail.com

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