COMPUTATIONAL RESULTS FOR SPECTRA OF HYPERGRAPHS

Aaron Dutle
Department of Mathematics
University of South Carolina

Abstract. We present a description of the algorithm used to compute the hypergraph spectra referred to in [2], and present tables of the characteristic polynomials computed, as well as timing data.

1. Introduction

Computing the multi-set spectrum of a k-uniform hypergraph (as defined in [2]) is a non-trivial task. Due to the characteristic polynomial's definition as the resultant of a system of n polynomials of degree $(k-1)$, the task is to compute the resultant. There are several very different algorithms known for doing such a computation, we refer the readers to [5, 1, 3] for three such algorithms. In our computations, we followed the algorithm in [3], adapted to our specific case of hypergraph spectra. We include a short description for completeness.

For $i \in [n]$, let $F_i = \sum_{e \in H(i)} x^e$ (following the notation in [2]). Each F_i is a degree $(k-1)$ homogeneous polynomial. Compute the characteristic polynomial as follows.

Let $d = n(k-2)+1$, and let S be the set of all monomials of degree d in the variables $x_1, \ldots x_n$. (We denote such a monomial x^α, where x stands for a variable vector, and α stands for an exponent vector.) Let

$$\begin{aligned} S_1 &= \{x^\alpha \in S \mid x_1^{k-1} \text{ divides } x^\alpha\} \\ S_2 &= \{x^\alpha \in S \setminus S_1 \mid x_2^{k-1} \text{ divides } x^\alpha\} \\ &\vdots \\ S_n &= \{x^\alpha \in S \setminus \bigcup_{i=1}^{n-1} S_i \mid x_n^{k-1} \text{ divides } x^\alpha\} \end{aligned}$$

This collection forms a partition of S (by an easy pigeon-hole principle argument). Fix an ordering on S, and make the $|S| \times |S|$ matrix M as follows. The (α, β) entry of M is the coefficient of x^β in the polynomial $F_i(x)\frac{x^\alpha}{x_i^{k-1}}$, where i is the unique index such that $x^\alpha \in S_i$. In particular, any non-zero (α, β) entry is one of the coefficients of F_i, where i has $x^\alpha \in S_i$.

Call a monomial $x^\alpha \in S$ *reduced* if there is exactly one i so that x_i^{k-1} divides x^α. Form the matrix M' be deleting the rows and columns of M that correspond to reduced monomials.

Then the characteristic polynomial of H is $\dfrac{\det(\lambda - M)}{\det(\lambda - M')}$.

2. Implementation

Our programs were written for the free and open-source SAGE mathematics program. Programs were written to convert between polynomials and hypermatrices, to create the polynomials that correspond to complete and complete tripartite hypergraphs, and several iterations and variations to compute the characteristic polynomial. All are available in the file `TensorCharpolyPackage.sage` which can be found at `www.math.sc.edu/~cooper/Resultant`.

The first working version of the program (called `tensor_charpoly_old` in the package) follows the algorithm in the introduction almost verbatim. The other variations incorporate a few time or space saving techniques. Some versions add features, such as allowing entries from any ring, or slight parallelization. Other versions are specialized in some way, one only allows 3-uniform hypergraphs, another only computes the characteristic polynomial for complete hypergraphs. It should be noted that the programs are written specifically for computing with hypergraphs, and although it will take any hypermatrix as an input, the output will only be valid for (weighted) adjacency hypermatrices.

We mention one particular technique (from [4]) we used to reduce time and space requirements because it is not well-known, and gave us significant gains. Given an $n \times n$ matrix A, one creates a digraph on $[n]$, with a directed edge (i, j) whenever the $a_{i,j}$ is nonzero. The characteristic polynomial of A can be computed as follows. For each strongly connected component C of the digraph, find characteristic polynomial of the A restricted to the rows and columns corresponding to vertices of C. The product of these polynomials is the characteristic polynomial of A. (Essentially, the strongly connected components give a good way to permute the rows and columns to obtain a block matrix.)

Computing the strongly connected components of a graph can be done in time $O(n^2)$, while most algorithms computing the characteristic polynomial run in time $O(n^3)$. Hence for large n, computing the strongly connected components adds little overhead, and if the strongly connected components are small, the savings in time and space to be had by computing the smaller characteristic polynomials can be great.

3. Computational Results

We performed computations on a desktop computer named "kiwi," with 8 cores (2× quad-core AMD at ? Ghz), and 8 giabytes of RAM. Timing statistics are computed using the function `cputime` in SAGE. For calculations that took less than 1 hour, the listed computation time is the best of 4 trials. Calculaions taking more than 1 hour were run only once. These calculations have their times marked with an asterisk. Horizontal breaks in a table indicate a change in the number of vertices.

Complete 3-cylinders
Computation performed on kiwi, using the function
`Ord3_tensor_charpoly`.

Part Sizes	Characteristic Polynomial	CPU Time (seconds)
1,1,1	$(x-1)^3x^3(x^2+x+1)^3$	0.030996
2,1,1	$x^{23}(x^3-4)^3$	0.128981
3,1,1	$(x-1)^9x^{44}(x^2+x+1)^9(x^3-9)^3$	0.634904
2,2,1	$x^{71}(x^3-16)^3$	0.628905
4,1,1	$x^{147}(x^3-16)^3(x^3-4)^{12}$	3.217511
3,2,1	$x^{156}(x^3-36)^3(x^3-4)^9$	3.408482
2,2,2	$(x-4)^3x^{183}(x^2+4x+16)^3$	3.662443
5,1,1	$(x-1)^{30}x^{304}(x^2+x+1)^{30}(x^3-25)^3(x^3-9)^{15}$	17.69331
4,2,1	$(x-4)^3x^{403}(x^2+4x+16)^3(x^3-16)^{12}$	20.204929
3,3,1	$(x-1)^{27}x^{304}(x^2+x+1)^{27}(x^3-81)^3(x^3-9)^{18}$	22.573569
3,2,2	$x^{412}(x^3-144)^3(x^3-16)^9$	27.308848
6,1,1	$x^{826}(x^3-36)^3(x^3-16)^{18}(x^3-4)^{45}$	111.613033
5,2,1	$x^{880}(x^3-100)^3(x^3-36)^{15}(x^3-4)^{30}$	148.966353
4,3,1	$x^{844}(x^3-144)^3(x^3-16)^9(x^3-36)^{12}(x^3-4)^{36}$	184.041021
4,2,2	$(x-4)^{12}x^{979}(x^2+4x+16)^{12}(x^3-256)^3$	265.255675
3,3,2	$x^{880}(x^3-324)^3(x^3-36)^{18}(x^3-4)^{27}$	341.33811
7,1,1	$(x-1)^{105}x^{1728}(x^2+x+1)^{105}(x^3-49)^3(x^3-25)^{21}(x^3-9)^{63}$	976.599535
6,2,1	$(x-4)^{18}x^{2106}(x^2+4x+16)^{18}(x^3-144)^3(x^3-16)^{45}$	1703.309058
5,3,1	$(x-1)^{90}x^{1728}(x^2+x+1)^{90}(x^3-225)^3(x^3-25)^9(x^3-81)^{15}(x^3-9)^{75}$	2259.604487
5,2,2	$x^{2160}(x^3-400)^3(x^3-144)^{15}(x^3-16)^{30}$	3991.632179*
4,4,1	$(x-4)^{24}x^{2079}(x^2+4x+16)^{24}(x^3-256)^3(x^3-16)^{48}$	2785.064607
4,3,2	$(x-4)^9x^{2124}(x^2+4x+16)^9(x^3-576)^3(x^3-144)^{12}(x^3-16)^{36}$	5808.092036*
3,3,3	$(x-9)^3(x-1)^{81}x^{1728}(x^2+9x+81)^3(x^2+x+1)^{81}(x^3-81)^{27}(x^3-9)^{81}$	11248.935901*
8,1,1	$(x-4)^3x^{4283}(x^2+4x+16)^3(x^3-36)^{24}(x^3-16)^{84}(x^3-4)^{168}$	12539.625686*
7,2,1	$x^{4544}(x^3-196)^3(x^3-100)^{21}(x^3-36)^{63}(x^3-4)^{105}$	23935.876193*
6,3,1	$x^{4328}(x^3-324)^3(x^3-144)^{18}(x^3-36)^{54}(x^3-16)^{54}(x^3-4)^{135}$	37704.312073*
6,2,2	$(x-4)^{45}x^{4922}(x^2+4x+16)^{45}(x^3-576)^3(x^3-256)^{18}$	72972.738457*
5,4,1	$x^{4400}(x^3-400)^3(x^3-100)^{12}(x^3-144)^{15}(x^3-16)^{30}(x^3-36)^{60}(x^3-4)^{120}$	59926.384805*

Complete 4-cylinders
Computation performed on kiwi, using the function
`tensor_charpoly`.

Part Sizes	Characteristic Polynomial	CPU Time (seconds)
1,1,1,1	$x^{44}(x^4-1)^{16}$	0.29337
1,1,1,2	$x^{213}(x^4-8)^{16}(x^4+1)^{32}$	4.93
1,1,1,3	$x^{1010}(x^4-27)^{16}(x^8+27)^{48}$	156.77
1,1,2,2	$x^{882}(x^4-1)^{64}(x^4-64)^{16}(x^4+8)^{64}$	155.90
1,1,1,4	$x^{3375}(x^4-1)^{192}(x^4+8)^{96}(x^4-64^{16}(x^8-20x^4+343)^{64}$	41,682.5*
1,1,2,3	$x^{3759}(x^4-216)^{16}(x^4+27)^{32}(x^8+1728)^{48}(x^8+27)^{96}$	43,230.1*
1,2,2,2	$x^{3401}(x^4+64)^{96}(x^4-512)^{16}(x^4+1)^{128}(x^4-8)^{192}$	31,253.14*

Complete 3-uniform Hypergraphs
Computation performed on kiwi, using the function
`charpoly_complete_hypergraph`.

Vertices	Characteristic Polynomial	CPU Time (seconds)
3	$(x-1)^3x^3(x^2+x+1)^3$	0.016997
4	$(x-3)x^4(x+1)^9(x-1)^{10}(x^2+x+4)^4$	0.06699
5	$(x-6)x^5(x-1)^{34}(x^2+9)^5(x^3+4x^2+7x+3)^{10}$	0.52692
6	$(x-10)x^6(x+2)^{30}(x-1)^{98}(x^2-2x+16)^6$ $(x^3+4x^2+13x+6)^{15}$	6.847959
7	$(x-15)x^7(x-1)^{258}(x^2-5x+25)^7$ $(x^3+3x^2+21x+10)^{21}(x^3+7x^2+19x+15)^{35}$	151.02404
8	$(x-21)x^8(x+3)^{105}(x-1)^{642}(x^2-9x+36)^8$ $(x^3+x^2+31x+15)^{28}(x^3+7x^2+28x+24)^{56}$	5536.073389*

Simplices (Complete k-uniform on $k+1$ vertices)
Computation performed on kiwi, using the function `charpoly_complete_hypergraph`.

Uniformity	Characteristic Polynomial	CPU Time (seconds)
3	$(x-3)x^4(x+1)^9(x-1)^{10}(x^2+x+4)^4$	0.066989
4	$(x-4)(x-1)^{15}(x+1)^{54}x^{100}$ $(x^3+x^2+7x+27)^5(x^3-x^2-8)^{20}$ $(x^4+4x^3+4x^2-8x-4)^{10}(x^4-2x^3+4x^2-2x+1)^{30}$	6.377031
5	$(x-5)(x+4)^{20}(x+1)^{120}(x-1)^{366}$ $x^{2118}(x^2-3x+1)^{90}(x^2+x+1)^{270}$ $(x^4+x^3+11x^2+51x+256)^6$ $(x^4+x^3+x^2-4x+16)^{120}$ $(x^5+5x^4+25x^3+35x^2+95x+27)^{15}$ $(x^6-x^5-10x^4+15x^3-70x^2-189x+729)^{30}$ $(x^6+2x^5+3x^4+8x^3+3x^2+2x+1)^{180}$ $(x^{16}-6x^{15}+25x^{14}-70x^{13}+210x^{12}-514x^{11}+689x^{10}$ $+230x^9-290x^8-2630x^7+7741x^6+234x^5+2085x^4$ $-4660x^3+2000x^2-576x+256)^{60}$	15000.894517*

Miscellaneous Hypergraphs
Computation performed on kiwi, using the function `Ord3_tensor_charpoly`.

Hypergraph	Characteristic Polynomial	CPU Time (seconds)
Q_3^2	$(x^3-1)^{18}(x^3-8)^{27}(x^3+1)^{54}x^{549}(x^3-2)^{486}$	4255.723333*
$K_4^{(3)}-e$	$x^{11}(x^3-12)(x^6-2x^3+5)^3$	0.131194

References

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[4] S.Lo, M. Monogan, and A. Wittkopf, *Strongly Connected Graph Components and Computing Characteristic Polynomials of Integer Matrices in Maple*, not in print, available at `http://www.cecm.sfu.ca/CAG/papers/CharPoly.pdf`, 2006.

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