

Problem Set 1: Solutions

MATH 778P, Fall 2008, Cooper

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You are awarded up to 12 points for each problem, 5 points for submitting solutions in L^AT_EX, and 5 points per solution that is used for the answer key.

1. For a set of integers S of cardinality n , define $\mu(S)$ by

$$\mu(S) = \max_{\alpha \in (0,1)} \min_{\substack{s,t \in S \\ s \neq t}} \|\alpha(s-t)\|,$$

where $\|x\| = \min_{r \in \mathbb{Z}} |x - r|$ is the distance to the nearest integer. Show that $\mu(S) > 1/n^2$.

Solution (Bill Kay): Choose $\alpha \in (0,1)$ randomly. Notice that $\|\alpha * x\| \pmod{1}$ is distributed uniformly on $[0,1]$ for $x \in \mathbb{Z} \setminus \{0\}$. For a given pair $s, t \in S$, Let $A_{s,t}$ be the event that $\|\alpha(s-t)\| \leq \frac{1}{n^2}$. Then $\Pr(A_{s,t} = \frac{2}{n^2})$. So now we need:

$$\Pr\left(\bigvee_{s,t \in S} A_{s,t}\right) \leq \binom{n}{2} \cdot \frac{2}{n^2} = \frac{n(n-1)}{2} < 1$$

So this implies $\Pr(\bigwedge_{s,t \in S} \overline{A_{s,t}} > 0)$, and the theorem follows.

2. Show that every graph on e edges has a subgraph on at least $e/2$ edges which is bipartite (i.e., 2-vertex-colorable).

Solution (Mark Walters): Suppose that $G = (V, E)$ is a graph with $|E| = e$. Randomly partition V into two sets X and Y , where

$$\Pr(v \in X) = \Pr(v \in Y) = \frac{1}{2}$$

for each $v \in V$. Let $A \subset E$ be the set of edges such that $\{u, v\} \in A$ if and only if $\{u, v\}$ meets both X and Y . Note that

$$\Pr(\{u, v\} \in A) = \frac{1}{2}$$

for every $\{u, v\} \in E$ because the endpoints are in opposite partition blocks in two of the four equally likely outcomes. It follows that

$$E[|A|] = \sum_{\{u,v\} \in E} \Pr(\{u, v\} \in A) = \frac{e}{2}.$$

Hence, there is some partition X, Y that realizes at least this expectation, i.e.,

$$\exists \text{ some partition } (X, Y) \text{ where } |A| \geq \frac{e}{2}.$$

Let H be the subgraph of G on the set of edges from A , which is a bipartite subgraph with at least $e/2$ edges.

3. Show that, for each n , there is a family \mathcal{F} of $m = \lfloor \frac{1}{2}(2/\sqrt{3})^n \rfloor$ subsets of $[n]$ so that no three distinct $A, B, C \in \mathcal{F}$ satisfy

$$A \cap B \subset C \subset A \cup B.$$

(Hint: Choose $2m$ sets, and then remove one set from each triple A, B, C that satisfies this condition.)

Solution (Mark Walters): Choose $2m$ subsets of $[n]$ randomly with uniform distribution for \mathcal{F} . For each $A \in \mathcal{F}$ let A be a $(0,1)$ -vector, where

$$a_i = \begin{cases} 1, & i \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Given distinct $A, B, C \in \mathcal{F}$, in order to force

$$A \cap B \subset C \subset A \cup B,$$

there are two implications that need to be satisfied for each $i \in [n]$:

1. if $c_i = 0$ then either $a_i = 0$ or $b_i = 0$
2. if $c_i = 1$ then either $a_i = 1$ or $b_i = 1$

Since there are two cases for each a_i and b_i , and the distribution is uniform, there are four equally likely possibilities for (a_i, b_i) , namely $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$. If $c_i = 0$ then the probability that implication 1 is satisfied is $3/4$ and if $c_i = 1$ then the probability that implication 2 is satisfied is $3/4$. So, given $A, B, C \in \mathcal{F}$,

$$Pr(A \cap B \subset C \subset A \cup B) = \left(\frac{3}{4}\right)^n.$$

Now, there are $2m$ ways to choose $C \in \mathcal{F}$. Given a choice of C , there are $\binom{2m-1}{2}$ ways to choose some $A, B \in \mathcal{F}$ both distinct from C and each other. Note that

$$\binom{2m-1}{2} < \binom{2m}{2} \leq \left(\left(\frac{2}{\sqrt{3}}\right)^n\right) < \frac{\left(\frac{4}{3}\right)^n}{2}.$$

Thus, there are fewer than $\frac{1}{2} \left(\frac{4}{3}\right)^n$ ways to select a triple A, B, C given $C \in \mathcal{F}$ and each triple has

$$Pr(A \cap B \subset C \subset A \cup B) = \left(\frac{3}{4}\right)^n.$$

Hence, given $C \in \mathcal{F}$, the probability that there exists $A, B \in \mathcal{F}$ distinct from C and each other such that $A \cap B \subset C \subset A \cup B$ is at most

$$\binom{2m-1}{2} \left(\frac{3}{4}\right)^n < \frac{1}{2} \left(\frac{4}{3}\right)^n \left(\frac{3}{4}\right)^n = \frac{1}{2}.$$

This means that there exists some family \mathcal{G} of $2m$ subsets of $[n]$ where fewer than half of the choices for $C \in \mathcal{G}$ will participate in a triple A, B, C such that $A \cap B \subset C \subset A \cup B$. Simply remove from \mathcal{G} all such sets C to form the family \mathcal{F} , which will have no such triples. Since $\frac{1}{2}(2m) = m$, we know that $|\mathcal{F}| \geq m$.

4. Show that there exists a tournament on n vertices with at least $n!/2^{n-1}$ Hamiltonian paths (i.e., directed paths that visit every vertex).

Solution (Wei-Tian Li): Consider a vertex sequence $v_{\pi(1)}v_{\pi(2)}\dots v_{\pi(n)}$, π is a permutation of $[n]$. Let X_π be the characteristic function that $v_{\pi(1)}v_{\pi(2)}\dots v_{\pi(n)}$ is a path in a random tournament. So $X_\pi = 1$ with probability $p = \frac{1}{2^{n-1}}$ since each $\pi(i)\pi(i+1)$ is an edge in the tournament with probability $\frac{1}{2}$. Therefore, the expectation of length n paths in a tournament is

$$E\left[\sum_{\pi} X_\pi\right] = \sum_{\pi} E[X_\pi] = \sum_{\pi} \frac{1}{2^{n-1}} = \frac{n!}{2^{n-1}}$$

Hence, there exists a tournament on n vertices with at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.

5. A k -SAT instance is a propositional formula of the form

$$S = C_1 \wedge \dots \wedge C_N$$

where each C_i has the form $a_1 \vee \dots \vee a_k$, and each a_k is a boolean variable or its negation. We say that S is “satisfiable” if there is an assignment of true or false to each variable so that S is true. Show that S is satisfiable if $N < 2^k$. Is this best possible?

Solution (Aaron Dutle): Assign the values true or false to each a_j with probability $1/2$. Let A_i be the event that C_i returns false. Notice that to return false for C_i , the assignment must return false for *each* a_j . Since there is only one way to do this of the 2^k possible assignments, $\Pr(A_i) = 1/2^k$. To return false for S , an assignment must return false for at least one C_i . Thus

$$\Pr(S \text{ false}) = \Pr\left(\bigvee_{i=1}^N A_i\right) \leq \sum_{i=1}^N \Pr(A_i) = N(1/2^k) < 1,$$

since $N < 2^k$. Thus $\Pr(S \text{ true}) > 0$, so there exists some assignment that satisfies S .

To see that this is the best possible, note that if $N = 2^k$ we can create an unsatisfiable instance as follows. For each T one of the 2^k subsets of $[k]$, let C_T use a_j for all $j \in T$ and use the negation of a_i for $i \notin T$. Then let $S = \bigwedge_{T \in 2^{[k]}} C_T$. Any assignment of truth values can be uniquely described by the indices that are set to false, giving some $T \subset [k]$. Then C_T returns false for this assignment, and this causes S to return false. Since this is true for any assignment of truth values, S always returns false, and so the bound is sharp.

6. Suppose a graph G on n vertices has average degree $d = 2|E(G)|/n$. Show that the size $\alpha(G)$ of the largest independent set is at least

$$\alpha(G) \geq \frac{n}{2d}.$$

(No using the Turán Theorem!)

Solution (Xing Peng): Let S be a random subset of G such that $\Pr\{v \in S\} = p$, where p is to be determined. Set $X = |S|$ and Y is the number of edges whose endpoints are in S . For each edge (u, v) , let $Y_{u,v}$ be the indicator random variable for the event $u, v \in S$. So $Y = \sum Y_{u,v}$ and $E[Y_{u,v}] = \Pr\{u, v \in S\} = p^2$. Hence, by linearity of expectation, $E[Y] = \sum_{(u,v) \in E(G)} Y_{u,v} = ep^2$, where e is the number of edges in G . Clearly, $E[X] = np$ and then $E[X - Y] = np - ep^2$. Let $p = 1/d$. Then $E[X - Y] = \frac{n}{2d}$. So we have a random subset S such that $E[X - Y] \geq \frac{n}{2d}$, which means the number of vertices in S is $\frac{n}{2d}$ greater than the number of edges in S . If we delete one endpoint of each edge in S , then the remaining subset is independent and has at least $\frac{n}{2d}$ vertices.

7. (Alon-Spencer #1) Prove that if there is a real p , $0 \leq p \leq 1$, so that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $r(k, t)$ satisfies $r(k, t) > n$. Conclude that $r(4, t) = \Omega(t^{3/2}/(\log t)^{3/2})$.

Solution (Wei-Tian Li): Coloring the edges of K_n randomly by the red color with probability p and by the blue color with probability $1 - p$. Let R be the event that there is a red clique with k vertices in K_n and B be the event that there is a blue clique with t vertices in K_n . The probability of no red K_k and blue K_t is

$$\begin{aligned} \Pr(\bar{R} \cap \bar{B}) &= \Pr(\overline{R \cup B}) \geq 1 - (\Pr(R) + \Pr(B)) \\ &= 1 - \left(\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} \right) > 0 \end{aligned}$$

provided that there is a real p , $0 \leq p \leq 1$, so that $\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1$. So there is a 2-coloring on the edges of K_n such that there is no red K_k nor blue K_t and hence $r(k, t) > n$.

For $r(4, t)$, we want to find p satisfying $\binom{n}{4}p^{\binom{4}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$. Assume $p = k \left(\frac{\log t}{t}\right)$, k to be determined. Let $n = \frac{1}{e} \left(\frac{t}{\log t}\right)^{3/2}$. The first term on the left side of the inequality is equal to

$$\begin{aligned} \binom{n}{4}p^{\binom{4}{2}} &= \frac{n(n-1)(n-2)(n-3)p^6}{24} \\ &< \frac{n^4 p^6}{24} = \frac{k^6}{24e^4}. \end{aligned}$$

The second term is equal to

$$\begin{aligned} \binom{n}{t}(1-p)^{\binom{t}{2}} &\leq \frac{n^t}{\sqrt{2\pi t} \left(\frac{t}{e}\right)^t} e^{-p \frac{t(t-1)}{2}} \\ &= \frac{1}{\sqrt{2\pi t}} \left(\frac{ne}{t} \cdot e^{-\frac{p(n-1)}{2}}\right)^t \\ &= \frac{1}{\sqrt{2\pi t}} e^{t(\log(\frac{ne}{t}) - \frac{p(n-1)}{2})} \\ &= \frac{1}{\sqrt{2\pi t}} e^{t\left(\log\left(\frac{\sqrt{t}}{(\log t)^{3/2}}\right) - \frac{k \log t}{t} \cdot \frac{t-1}{2}\right)}. \end{aligned}$$

Set $k = 1$, then the exponent is equal to

$$\begin{aligned} &\log\left(\frac{\sqrt{t}}{(\log t)^{3/2}}\right) - \frac{k \log t}{t} \cdot \frac{t-1}{2} \\ &= \frac{1}{2} \log t - \log(\log t)^{3/2} - \frac{\log t}{t} \cdot \frac{t-1}{2} \\ &= -\log(\log t)^{3/2} + \frac{\log t}{t}. \end{aligned}$$

Let t approach positive infinity, then the exponent approaches negative infinity. Hence the limit of the second term is zero. So take $p = \frac{\log t}{t}$ and $n = \frac{1}{e} \left(\frac{t}{\log t}\right)^{3/2}$, then $\binom{n}{4}p^{\binom{4}{2}} + \binom{n}{t}(1-p)^{\binom{t}{2}} < 1$ and $r(4, t) = \Omega\left(\left(\frac{t}{\log t}\right)^{3/2}\right)$.

8. (Alon-Spencer #2) Suppose $n \geq 4$ and let H be an n -uniform hypergraph (i.e., a set family whose elements have cardinality n) with at most $4^{n-1}/3^n$ edges. Prove that there is a coloring of the vertices of H by four colors so that, in every edge, all four colors are used.

Solution (Yiting Yang): Let E be the set of edges of H . Color each vertex of H randomly and uniformly with four colors. Let A_e be the event that vertices in the edge e were colored by at most three colors.

We have 4 ways to select 3 colors to color the vertices of e and for each selection, there are 3^n different colorings. So $Pr(A_e) \leq \frac{4 \cdot 3^n}{4^n} = \frac{3^n}{4^{n-1}}$

Since A_e and A_f are not disjoint when e and f are not disjoint,

$$Pr(\cup_{e \in E} A_e) < \sum_{e \in E} Pr(A_e) < \frac{4^{n-1}}{3^n} \frac{3^n}{4^{n-1}} = 1$$

which implies

$$Pr(\cap_{e \in E} \overline{A_e}) > 0.$$

So there is a coloring of the vertices of H by four colors so that, in every edge, all four colors are used.

9. (Alon-Spencer #5) Let G be a graph on $n \geq 10$ vertices and suppose that if we add to G any edge not in G , then the number of copies of K_{10} increases. Show that the number of edges of G is at least $8n - 36$.

Solution (Aaron Dutle): Let E be the set of non-edges of G (the edges of \overline{G}). For each edge $e \in E$, build a pair of sets (A_e, B_e) as follows.

Let B_e be the endvertices of the edge e . For A_e , choose one of the copies of K_{10} that is created when e is added to G , and let A_e be all the vertices of G not used in this K_{10} .

Note that $|A_e| = n - 10$ and $|B_e| = 2$ for each edge. Notice that $A_e \cap B_e = \emptyset$, since the vertices of B_e are part of the K_{10} that was left out to create A_e .

Claim that if $f \neq e$, then $A_f \cap B_e \neq \emptyset$. Suppose it were empty. Then both vertices of e are in some K_{10} that was created by adding the edge f . This means that e must have already been in the graph (otherwise the K_{10} is missing this edge). This contradicts that e is a non-edge, proving the claim.

Thus $\{(A_e, B_e)\}_{e \in E}$ is an $(n - 10, 2)$ system, (as in Theorem 1.3.3 in the text). So by Bollobás' result, $|E| \leq \binom{n-8}{2}$.

Since E is the number of non-edges in G , the number of edges of G is

$$\binom{n}{2} - |E| \geq \binom{n}{2} - \binom{n-8}{2} = \frac{n^2 - n - n^2 + 17n - 72}{2} = 8n - 36,$$

our desired result.

10. (Alon-Spencer #10) Prove that there is an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length $c\sqrt{n}$.

Solution (Bill Kay): Let $\pi : \mathbb{N} \rightarrow \mathbb{N}$ be a permutation of $[n]$. We want to calculate the probability that π has an increasing subsequence of length k . First, notice that there are $\binom{n}{k}$ ways to pick the k elements of π which could be in the subsequence. For any k elements of π , the sequence is increasing with probability of $\frac{1}{k!}$, since just one out of the $k!$ arrangements of the k elements is monotone increasing. This gives us that the probability that π contains a monotone increasing subsequence is $\binom{n}{k} \frac{1}{k!}$. To simplify the asymptotics, we apply Stirling's approximation and get:

$$\binom{n}{k} \frac{1}{k!} \leq \left(\frac{en}{k}\right)^k \cdot \frac{1}{\left(\frac{k}{e}\right)^k}$$

$$= \left(\frac{ne^2}{k^2} \right)^k.$$

We now make the substitution $k = c\sqrt{n}$:

$$\left(\frac{ne^2}{c\sqrt{n}^2} \right)^{c\sqrt{n}} = \left(\frac{e^2}{c^2} \right)^{c\sqrt{n}},$$

which is clearly less than 1 for any value of $c > e$. Let A_i be the event that row i has a monotone increasing subsequence of length $c\sqrt{n}$ with $c = e + o(1)$. We have already seen $P(A_i) < 1$. Let $c = e\sqrt{2}$. Further, let $A = \sum_{i \leq n} A_i$. Then $\Pr(A) \leq n \cdot \left(\frac{1}{2}\right)^{e\sqrt{2}\sqrt{n}}$, which is clearly less than 1. So there exists some permutation in which there is no such subsequence in any of the rows.