

Problem Set 1

MATH 778P, Fall 2008, Cooper

Expiration: Thursday September 15

You are awarded up to 12 points for each problem, 5 points for submitting solutions in L^AT_EX, and 5 points per solution that is used for the answer key.

1. For a set of integers S of cardinality n , define $\mu(S)$ by

$$\mu(S) = \max_{\alpha \in (0,1)} \min_{\substack{s,t \in S \\ s \neq t}} \|\alpha(s-t)\|,$$

where $\|x\| = \min_{r \in \mathbb{Z}} |x-r|$ is the distance to the nearest integer. Show that $\mu(S) > 1/n^2$.

2. Show that every graph on e edges has a subgraph on at least $e/2$ edges which is bipartite (i.e., 2-vertex-colorable).
3. Show that, for each n , there is a family \mathcal{F} of $m = \lfloor \frac{1}{2}(2/\sqrt{3})^n \rfloor$ subsets of $[n]$ so that no three distinct $A, B, C \in \mathcal{F}$ satisfy

$$A \cap B \subset C \subset A \cup B.$$

(Hint: Choose $2m$ sets, and then remove one set from each triple A, B, C that satisfies this condition.)

4. Show that there exists a tournament on n vertices with at least $n!/2^{n-1}$ Hamiltonian paths (i.e., directed paths that visit every vertex).
5. A k -SAT instance is a propositional formula of the form

$$S = C_1 \wedge \cdots \wedge C_N$$

where each C_i has the form $a_1 \vee \cdots \vee a_k$, and each a_k is a boolean variable or its negation. We say that S is “satisfiable” if there is an assignment of true or false to each variable so that S is true. Show that S is satisfiable if $N < 2^k$. Is this best possible?

6. Suppose a graph G on n vertices has average degree $d = 2|E(G)|/n$. Show that the size $\alpha(G)$ of the largest independent set is at least

$$\alpha(G) \geq \frac{n}{2d}.$$

(No using the Turán Theorem!)

7. (Alon-Spencer #1) Prove that if there is a real p , $0 \leq p \leq 1$, so that

$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$

then the Ramsey number $r(k, t)$ satisfies $r(k, t) > n$. Conclude that $r(4, t) = \Omega(t^{3/2}/(\log t)^{3/2})$.

8. (Alon-Spencer #2) Suppose $n \geq 4$ and let H be an n -uniform hypergraph (i.e., a set family whose elements have cardinality n) with at most $4^{n-1}/3^n$ edges. Prove that there is a coloring of the vertices of H by four colors so that, in every edge, all four colors are used.
9. (Alon-Spencer #5) Let G be a graph on $n \geq 10$ vertices and suppose that if we add to G any edge not in G , then the number of copies of K_{10} increases. Show that the number of edges of G is at least $8n - 36$.
10. (Alon-Spencer #10) Prove that there is an absolute constant $c > 0$ with the following property. Let A be an $n \times n$ matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length $c\sqrt{n}$.