Problem Set 1

MATH 777, Spring 2008, Cooper

Expiration: Friday February 15

Each problem is worth 20 points. Rigorous proofs are required for all claims, although elegance and concision are nearly as important. You may only use results we proved in class, or which can reasonably be considered prerequisite material for this class, unless otherwise stated.

1. A graph G is called "uniquely colorable" if it has exactly one partition into $\chi(G)$ independent sets, i.e., a $\chi(G)$-coloring is unique up to permutation of the colors. Show that a graph is uniquely 2-colorable iff it is bipartite and connected.

2. Suppose G is uniquely k-colorable with proper k-coloring $c : V(G) \to [k]$. Define $a_j = 2^{|c^{-1}(j)|} - 1$. Show that there are at least

$$\prod_{j=1}^{k} a_j \cdot \sum_{j=1}^{k} a_j^{-1}$$

uniquely k-colorable graphs G' with $|V(G')| = |V(G)| + 1$ and $G \subset G'$.

3. Diestel §5, #16 (#17 in 3^{rd} edition). What is $P_G(\lambda)$ for $G = K_t$ or any tree on t vertices?

4. Compute $P_G(\lambda)$ for $G = C_n$ and $W_n = K_1 * C_n$.

5. Let G have vertices which are intervals of $\mathbb{R}$, and $x \sim y$ iff $x \cap y \neq \emptyset$. (Such a graph is called an "interval graph," for obvious reasons.) Show that G is chordal, and therefore perfect. Is this statement still true if the vertices of G are unions of two intervals?

6. For which n is the wheel graph $W_n = K_1 * C_n$ perfect?