Final Exam

MATH 776, Fall 2007, Cooper

Thursday December 13

Problems are ranked based on their difficulty, indicated in parentheses next to the problem number. The number of points awarded for a **fully correct, rigorous** proof is indicated next to each problem. (Note that this is different than the scoring system on the problem sets.) It is very important that your writing be legible and your arguments concise and elegant – so think before you commit anything to paper. You may use any theorem covered in class, and any result that one could reasonably expect to know from an undergraduate mathematics education. (If you are unsure about something, ask.)

Some definitions:

1. The **undirected Cayley graph** of (Γ, S) with $S = S^{-1} \subset \Gamma$ has vertex set Γ, with $x \sim y$ if $\exists \gamma \in S$ so that $x = \gamma y$.
2. The automorphism group $\mathrm{Aut}(G)$ **acts transitively** on $V(G)$ if, for all $x, y \in V(G)$, $\exists f \in \mathrm{Aut}(G)$ so that $f(x) = y$. In this case, we say that G is **vertex-transitive**.
3. The **adjacency matrix** of a graph H is the $|V(H)| \times |V(H)|$ matrix A indexed by the vertices of H so that $A_{v,w} = 1$ if $v \sim w$ and 0 otherwise. The **incidence matrix** of a graph H is the $|E(G)| \times |V(H)|$ matrix B indexed by edges and vertices so that $B_{e,v} = 1$ if $v \in e$ and 0 otherwise.
4. The size of the smallest independent set in a graph G is denoted $\alpha(G)$, and the size of the largest clique (complete graph) in G is denoted $\omega(G)$.
5. A graph G is called **perfect** if, for each induced $H \subset G$, $\chi(H) = \omega(H)$.
6. The **Kneser graph** $KG_{n,k}$ has vertex set $\binom{[n]}{k}$, with $S \sim T$ iff $S \cap T = \emptyset$.
7. A **chordal graph** is a graph in which every induced cycle has at most three vertices.
8. The **Wheel graph** W_n is $C_n * K_1$.
9. The square of a graph G, denoted G^2, has the same vertex set as G, with $xy \in E(G^2)$ iff $xy \in E(G)$ or $\exists z : (xz \in E(G)) \wedge (zy \in E(G))$.

10. A **dominating set** $S \subset V(G)$ has the property that, for all $v \in V(G)$, $v \in S$ or $v \sim S$. The size of the smallest dominating set is denoted $\gamma(G)$.

11. A graph G is a **cograph** if $G = K_1$, G is the complement of a cograph, or G is the disjoint union of two cographs.

12. An order n (binary) **de Bruijn cycle** is a binary sequence $(a_0, \ldots, a_{2^n})$ so that, for each $\mathbf{b} \in \{0,1\}^n$, there is exactly one j, $1 \leq j \leq 2^n$, so that $(a_j, \ldots, a_{j+n-1}) = \mathbf{b}$, with indices considered modulo 2^n. That is, each binary n-word occurs in the sequence exactly once as consecutive symbols (possibly with wrap-around).

13. The **chromatic polynomial** $P_G(k)$ of a graph G is the number of proper vertex k-colorings.

And the problems:

1. (7) Prove that the line graph $L(G)$ of any graph G is "claw-free," i.e., contains no induced $K_{1,3}$.

2. (5) Show that, for any graph G,

$$\|L(G)\| + |L(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v)^2.$$

3. (12) Use problem 2 to show that, for any planar triangulation G,

$$\|L(G)\| \geq 15n - 65.$$

(Hint: The relevant version of the Cauchy-Schwarz Inequality says that, for $\mathbf{a} \in \mathbb{R}^n$, $n \sum a_i^2 \geq (\sum a_i)^2$.)

4. (9) Show that $C_{100} \Box C_{100}$ is not planar.

5. (5) Prove that every undirected Cayley graph is vertex-transitive.

6. (15) Prove that the line graph of a bipartite graph is perfect.

7. (5) Prove that every bipartite graph is perfect.

8. (12) Prove that the complement of a bipartite graph is perfect. (Hint: König's Theorem.)

9. (5) If $L(G) = W_4$, then what is G?

10. (8) Show that the incident matrix $\mathcal{L}$ of $L(G)$ is related to the adjacency matrix $\mathcal{B}$ of G by

$$\mathcal{L} = \mathcal{B}^* \mathcal{B} - 2I.$$

11. **(13)** Prove that the eigenvalues of the adjacency matrix of the hypercube Q_n are precisely $n - 2k$, $k = 0, \dots, n$, with multiplicity $\binom{n}{k}$.

12. **(6)** Give an expression for the number of closed walks of length t in Q_n. (Hint: Use the previous problem, and consider the trace of powers of the adjacency matrix.)

13. **(10)** Suppose that G is regular of odd degree k, and $\lambda(G) \geq k - 1$. Show that G contains a 1-factor.

14. **(11)** Suppose that G is cubic planar, with a C_6-faces and b C_r-sided faces. Show that $(b, r) = (4, 3)$, $(6, 4)$, or $(12, 5)$.

15. **(8)** Prove the reconstruction conjecture for regular graphs, that is, if G is regular and H is some graph not isomorphic to G, then the multisets $\{G - v : v \in V(G)\}$ and $\{H - v : v \in V(H)\}$ are different.

16. **(5)** Prove that a graph is regular if and only if the all-ones vector (i.e., $(1, 1, \dots, 1, 1)$) is an eigenvector of its adjacency matrix. What is the corresponding eigenvalue?

17. **(6)** Prove that the Kneser graph $KG_{n,k}$ contains a (not necessarily induced) $K_{r,r}$ with

$$r = \binom{\lfloor n/2 \rfloor}{k}.$$

18. **(14)** Show that, if $\chi(G) \geq k$, then G contains *every* tree on $k + 1$ vertices. (Hint: G contains a subgraph H with large minimum degree.)

19. **(12)** Suppose that x, y are central vertices in G, and z maximizes $d(x, z)$. If P is a x–z path of minimum length, show that $d(y, P) < r/2$.

20. **(7)** Show that the wheel graph is self-dual, i.e., it is its own *unique* dual graph.

21. **(12)** Let G be the complement of C_n. For which n is G planar?

22. **(5)** For a binary operator $* : (G_1, G_2) \mapsto G_3$, define $G^{*1} = G$ and $G^{*k} = G^{*(k-1)} * G$ for $k > 1$. How many edges does $G^{\square k}$ have? And $\|G^{\times k}\|$?

23. **(12)** Suppose G is chordal. Show that G contains a clique of size at least $\kappa(G)$. (Hint: Menger.)

24. **(5)** How many paths of length k are there in the complete graph K_n?

25. **(5)** Prove that, if G_1 and G_2 are vertex-transitive, then $G_1 \square G_2$ is vertex-transitive.

26. **(8)** Prove that $\gamma(G) \leq \alpha(G)$.

27. **(10)** Let M be a 1-factor of $K_{n,n}$. Prove that the number of 1-factors of $K_{n,n} - M$ is $n!/e \pm 1$. (Hint: Inclusion-exclusion.)

28. **(8)** Prove that if G is a cograph, then G has no induced path of length 3.

29. **(6)** Prove that $KG_{n,k}$ is connected iff $n > 2k$.

30. **(11)** Prove that the number U_n of unicyclic connected labeled graphs on n vertices satisfies

$$\binom{n-1}{2} n^{n-3} \leq U_n \leq \binom{n-1}{2} n^{n-2}.$$

31. **(13)** Prove that, for G of diameter at least 3, $\chi(G^2) \leq \Delta(G)^2$.

32. **(6)** Prove that, if G is Hamiltonian, then $\gamma(G) \leq \lceil |G|/3 \rceil$.

33. **(7)** What is $\gamma(C_n)$?

34. **(6)** Show that every undirected Cayley graph is Type I.

35. **(7)** Show that the set of all forests is minor-closed, and that every non-forest contains K_3 as a minor.

36. **(12)** Prove that every graph G with $|G| = 2n$ and $\delta(G) \geq n$ contains a 1-factor. (Hint: If not, then consider a pair u, v of vertices not covered by a maximum matching M. If $\{x, y\} \in M$, can $E(\{u, v\}, \{x, y\}) \geq 3$?)

37. **(10)** Let T be a tree, and define the graph G to have vertices which are subtrees of T, with $T_1 \sim_G T_2$ iff $T_1 \cap T_2 \neq \emptyset$. Show that G is chordal.

38. **(5)** Show that every G with $\|G\| > 0$ possesses a partition $V(G) = V_1 \dot\cup V_2$ into two nonempty, disjoint sets so that $\chi(G[V_1]) + \chi(G[V_2]) = \chi(G)$.

39. **(7)** Show that, for any two graphs G_1 and G_2, $\chi(G_1 \cup G_2) \leq \chi(G_1)\chi(G_2)$.

40. **(14)** Show that, if one could compute $\alpha(G)$ in time polynomial in $|G|$ for any G, then one could approximate the size of the largest induced cycle of H within a factor of 3 in time polynomial in $|H|$ for any H.

41. **(10)** Prove that any k-vertex-connected subgraph of an edge-minimal k-vertex-connected graph is itself edge-minimally k-vertex-connected.

42. **(8)** Show that a binary de Bruijn cycle exists for each order $n \geq 1$ by considering the digraph B_n with $V(B_n) = \{0, 1\}^{2^n}$ and $x \sim y$ iff

$$(x_1, \ldots, x_{n-1}) = (y_0, \ldots, y_{n-2}).$$

43. **(7)** What is $P_G(k)$ for $G = K_n$? If G is an independent set of size n? If G is a tree on n vertices?

44. **(10)** Show that $\lim_{k \to \infty} P_G(k) k^{-|G|} = 1$.

45. **(10)** Prove that every graph G satisfies $\alpha(G) \geq |G|/(\Delta(G)+1)$. (Hint: Order the vertices randomly, then greedily select an independent set. What is its expected size?)