Iterated Point-Line Configurations Grow Doubly-Exponentially

Joshua Cooper and Mark Walters

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Abstract

Begin with a set of four points in the real plane in general position. Add to this collection the intersection of all lines through pairs of these points. Iterate. Ismailescu and Radoičić (2003) showed that the limiting set is dense in the plane. We give doubly exponential upper and lower bounds on the number of points at each stage. The proof employs a variant of the Szemerédi-Trotter Theorem and an analysis of the "minimum degree" of the growing configuration.

Consider the iterative process of constructing points and lines in the real plane given by the following: begin with a set of points $P_1 = \{p_1, p_2, p_3, p_4\}$ in the real plane in general position. For each pair of points, construct the line passing through the pair. This will create a set of lines $L_1 = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6\}$. Some of these constructed lines will intersect at points in the plane that do not belong to the set P_1. Add any such point to the set P_1 to get a new set P_2. Now, note that there exist some pairs of points in P_2 that do not lie on a line in L_1, namely some elements of $P_2 \setminus P_1$. Add these missing lines to the set L_1 to get a new set L_2. Iterate in this manner, adding points to P_k followed by adding lines to L_k. We assume that the original configuration is such that for every $k \in \mathbb{N}$ no two lines in L_k are parallel.

Now we introduce some notation for this iterative process. The k^{th} *stage* is defined to consist of these two ordered steps:

1. Add each intersection of pairs of elements of L_k to P_{k+1}, and

2. Add a line through each of pair of elements of P_k to L_{k+1}.

Under this definition, we say that stage 1 begins with the configuration of four points with six lines and stage k begins with n_k points with m_k lines. We will denote the set of points at the beginning of stage k by P_k and likewise the set of lines at the beginning of stage k by L_k. There are some trivial bounds on the number of points and lines at stage k that can be obtained with this notation. Since a point in P_k must lie at the intersection of at least two lines of L_{k-1} we know that at stage k, there are at most $\binom{m_{k-1}}{2}$ points. Similarly, since a line in L_k must contain at least two points from P_k we know that at stage k there are at most $\binom{n_k}{2}$ lines. In other words,

$$n_k \leq \binom{m_{k-1}}{2} \qquad \text{and} \qquad m_k \leq \binom{n_k}{2}.$$

From this it follows that

$$n_{k+1} \leq \binom{m_k}{2} \leq \binom{\binom{n_k}{2}}{2} < \binom{\frac{{n_k}^2}{2}}{2} < \frac{\left(\frac{{n_k}^2}{2}\right)^2}{2} = \frac{{n_k}^4}{8}$$

and

$$m_{k+1} \leq \binom{n_{k+1}}{2} \leq \binom{\binom{m_k}{2}}{2} < \binom{\frac{{m_k}^2}{2}}{2} < \frac{\left(\frac{{m_k}^2}{2}\right)^2}{2} = \frac{{m_k}^4}{8}.$$

Note that a stage in this iterative process can be alternatively defined as follows:

1. Place a point at any intersection of a pair of lines for which a point does not already exist.

2. Take the dual of the configuration of points and lines (points become lines and lines become points).

3. Return to step 1.

Hence, points and lines play a very similar role in this process and we only need to consider bounds on one of the two quantities. Henceforth we will only provide arguments concerning the bounds on n_k. A trivial lower bound is given in the following:

Proposition 1. *For all $k \in \mathbb{N}$, $n_{k+1} \geq n_k + 1$.*

Proof. If this claim is false then we must have a stage at which the process stabilizes [1]. So, suppose that the process stabilizes at the beginning of stage k and let $\mathrm{conv}(P_k)$ denote the convex hull of P_k, where $|\mathrm{conv}(P_k)|$ denotes the number of vertices of this convex hull. Suppose first that $|\mathrm{conv}(P_k)| \geq 4$. In this case, we can find two nonadjacent, nonparallel sides of the convex hull, which lie on lines that intersect outside of the convex hull. This contradicts the stability supposition. So, $|\mathrm{conv}(P_k)| = 3$. Let $\{a, b, c\}$ be the set of vertices of the triangle forming the convex hull. Suppose that there exist points along at least two of the sides of the triangle defined by $\{a, b, c\}$, say $x \in ab$ and $y \in bc$. In this case, the line formed by xy must intersect ac outside the convex hull, again contradicting stability. So, there exist points along at most one of the sides of the triangle defined by $\{a, b, c\}$. Suppose that there exists some point x in the interior of $\{a, b, c\}$ and define $y = ax \cap bc$, $z = cx \cap ab$. In this case, we have $y \in bc$ and $z \in ab$, a contradiction to the assumption that at most one side of the triangle contains points. The only remaining possibility is that P_k is comprised of $n_k - 1$ collinear points. But, the starting configuration of points and lines has the condition that for any line in L_1, there are at least two points of P_1 not passing through it. Since we never remove any points during this process, then this must hold true for every stage, in particular stage k. This contradiction completes the proof. $\square$

We define the *degree* of a point $p \in P_k$, denoted $d_k(p)$, to be the number of distinct lines incident upon p at the beginning of stage k. Similarly, the degree of a line $\ell \in L_k$, denoted $d_k(\ell)$, is the number of distinct points through which it passes at the beginning of stage k. Also, let

$$\delta_k = \min\{d_k(p) \mid p \in P_k\} \qquad \text{and} \qquad \overline{\delta}_k = \min\{d_k(\ell) \mid \ell \in L_k\}$$

and

$$\Delta_k = \max\{d_k(p) \mid p \in P_k\} \qquad \text{and} \qquad \overline{\Delta}_k = \max\{d_k(\ell) \mid \ell \in L_k\}.$$

Define an $n \times n$ *grid* to be any configuration of two collections of n parallel lines, where the one collection is not parallel to the other. Using these definitions, we obtain the following observation:

Proposition 2. *For all* $k \in \mathbb{N}$, $\delta_k \geq 3$.

Proof. Suppose to the contrary that there exists some $k \in \mathbb{N}$ with $\delta_k < 3$. Since there are no points of degree 1, we must have $\delta_k = 2$. So there exists $p \in P_k$ with $d_k(p) = 2$, i.e., there exist two lines $\ell, \ell' \in L_k$ with $P_k \subseteq \ell \cup \ell'$. Note that $n_2 = 7$ and $\overline{\Delta}_2 = 3$ and so for all $\ell, \ell' \in L_2$, $P_2 \nsubseteq \ell \cup \ell'$. Since we never remove points in this iterative process, we know that if there exists $\ell, \ell' \in L_k$ with $P_k \subseteq \ell \cup \ell'$, then $k < 2$, i.e., $k = 1$. But we know that $\delta_1 = 3$, a contradiction. $\square$

We can obtain major improvements to the trivial lower bound using the following:

Lemma 3. *The minimum number of parallel lines required to pass through all of the intersections of an $n \times n$ grid is $2n - 1$.*

Proof. Suppose that Q and R are sets of parallel lines that comprise an $n \times n$ grid. Let S be a minimal witness set of s parallel lines passing through all intersections of the grid. We aim to show that $s \geq 2n - 1$. Without loss of generality, orient the grid so that the lines of S are vertical in the xy-plane and let $X = \{x_1, x_2, \ldots, x_s\}$ be the x-intercepts of the lines of S. So X is the collection of projected points, when we project the grid intersections onto the x-axis with this orientation. Let $\pi(p)$ denote the projection of a point p in the grid onto the x-axis. Arbitrarily choose lines ℓ_q, ℓ_r in the grid with $\ell_q \in Q$ and $\ell_r \in R$. Let $q_1, q_2, \ldots, q_n$ and $r_1, r_2, \ldots, r_n$ be the points of intersection of ℓ_q with R and ℓ_r with Q, respectively, where

$$\pi(q_1) \leq \pi(q_2) \leq \cdots \leq \pi(q_n)$$

and

$$\pi(r_1) \leq \pi(r_2) \leq \cdots \leq \pi(r_n).$$

Suppose also that $q_i = r_j$. Define A and B to be the sets of real numbers given by

$$A = \{\pi(q_1), \pi(q_2), \ldots, \pi(q_n)\}$$

and

$$B = \{\pi(r_1) - \pi(r_j), \pi(r_2) - \pi(r_j), \ldots, \pi(r_n) - \pi(r_j)\}.$$

Under this setting we have that $S = A + B$ and thus

$$s = |A + B|.$$

It is well known that

$$|A+B| \geq 2n-1$$

for any pair A, B of sets of cardinality n and that equality is achieved when A and B are arithmetic progressions [2]. It follows that $s \geq 2n-1$, completing the proof. $\square$

Using this lemma we can prove the following:

Theorem 4. $\delta_{k+1} \geq \min\{n_k - 1, 2\delta_k - 3\}$.

Proof. Let $p \in P_k$. It suffices to show that

$$d_{k+1}(p) \geq \min\{n_k - 1, 2\delta_k - 3\}.$$

First suppose each line in L_k that passes through p has degree 2. In this case, it's easy to see that there are $d_k(p)+1$ points at the beginning of stage k and so $d_k(p) = n_k - 1$. Since we never remove lines, we know that

$$\begin{aligned} d_{k+1}(p) &\geq d_k(p) \\ &= n_k - 1 \\ &\geq \min\{n_k - 1, 2\delta_k - 3\}. \end{aligned}$$

Now suppose there exists a line $\ell \in L_k$ that passes through p with $d_k(\ell) \geq 3$. Let $q, r \in P_k$ be the other two points on ℓ. Note that $d_k(q) \geq \delta_k$ and $d_k(r) \geq \delta_k$ and so there exist two sets of lines

$$L_q = \{\ell_{q_1}, \ell_{q_2}, \ldots, \ell_{q_n}\} \subseteq L_k \backslash \ell \quad \text{and} \quad L_r = \{\ell_{r_1}, \ell_{r_2}, \ldots, \ell_{r_m}\} \subseteq L_k \backslash \ell,$$

where $n, m \geq \delta_k - 1$ and the sets $L_q \cup \ell$ and $L_r \cup \ell$ consist of the lines incident upon q and r, respectively. Now, consider the real plane as a subset of the real projective plane in the standard way and let ℓ be the line at infinity. We restrict our attention to arbitrarily chosen subsets ${L_q}' \subseteq L_q$ and ${L_r}' \subseteq L_r$, where $|{L_q}'| = |{L_r}'| = \delta_k - 1$. These lines form a $(\delta_k - 1) \times (\delta_k - 1)$ grid. Now in this grid we will place a point at each intersection for which one does not already exist during stage k. After doing so, we will construct a line through each pair of points for which one does not already exist. In particular, we will do so for pairs of points of the form (p, x), where x lies at the intersection of lines from ${L_q}'$ and ${L_r}'$. So, at the beginning of stage $k+1$, there will be at least s lines incident upon p, where s denotes the number of lines necessary to adjoin p with all of the intersections of the grid. In other words, $d_{k+1}(p) \geq s$. Note that any lines passing through p would form a third collection of parallel lines to add to the grid. Therefore, s is at least the minimum number of parallel lines required to pass through all of the intersections of a $(\delta_k - 1) \times (\delta_k - 1)$ grid. Applying Lemma 1 yields

$$\begin{aligned} d_{k+1}(p) &\geq s \\ &\geq 2(\delta_k - 1) - 1 \\ &= 2\delta_k - 3 \\ &\geq \min\{n_k - 1, 2\delta_k - 3\}. \end{aligned}$$

□

Now by using techniques similar to the preceding proofs, we can obtain even faster growth of the minimum degree. We will then use the growth rate of δ_k to provide arguments for a better lower bound on n_k. First, let $cr(G)$ denote the crossing number of a graph, which is the minimum number of crossings in a planar drawing of the graph G. We will use the following lemma regarding crossing numbers (the proof can be found in [3]):

Lemma 5. *If a graph G with n vertices and e edges has $e > 7.5n$, then we have*

$$cr(G) \geq \frac{e^3}{33.75n^2}.$$

We now use this crossing number inequality in the following theorem. The argument closely resembles Székely's proof ([4]) of the Szemerédi-Trotter Theorem (first appearing in [5]).

Theorem 6. *Let $\mathcal{F} = \{F_1, F_2, \ldots, F_N\}$ be a collection of $N \geq 4$ families, each of exactly $k \geq 2$ parallel lines, no two collections parallel to each other. Let P denote the collection of points that lie at the intersections of lines ℓ_i and ℓ_j, where $\ell_i \in F_1$ and $\ell_j \in F_j$ for some $2 \leq j \leq k$. Then*

$$|P| \geq ck^2 N^{1/2},$$

where c is a positive real constant.

Proof. Let A denote this configuration of $|P|$ points and Nk lines. Let i be the number of point-line incidences in A. Note that there are N different families of parallel lines in A, each containing exactly k lines. For all families except F_1, each line contains exactly k points from P and thus contains exactly $k-1$ line segments which connect two points from A, call them edges. We know that $k \geq 2$ and so $k-1 \geq k/2$. Hence, each line contains at least $k/2$ edges and if we add this up over all of the Nk lines, we see that the number of edges obtained in this manner is at least half of the total number of incidences. In other words,

$$\text{(total number of edges)} \geq \frac{i}{2}.$$

Now, we can count the exact number of edges in A. For the k lines of F_1, there are $|P|-k$ edges because all $|P|$ points lie on the lines of F_1 and for each line we must subtract one to count the number of edges. For each of the remaining $N-1$ families there are exactly k lines, each containing exactly $k-1$ edges, yielding a total of

$$(N-1)k(k-1)$$

edges. Adding these quantities together, we obtain a grand total of

$$|P|-k+(N-1)k(k-1)$$

edges, which simplifies to

$$|P|+Nk(k-1)-k^2.$$

Now consider the graph G with $V(G)=P$ and $E(G)$ consisting of the aforementioned edges. Since all of the edges lie on one of Nk lines, and any two lines intersect in at most one point, we have

$$\mathrm{cr}(G) \leq (Nk)^2.$$

Applying the crossing number inequality, we obtain that either

$$|P|+Nk(k-1)-k^2 \leq 7.5|P| \tag{1}$$

or that

$$(Nk)^2 \geq \frac{(|P|+Nk(k-1)-k^2)^3}{33.75|P|^2} \tag{2}$$

In the case of (1) we get

$$Nk(k-1) - k^2 \leq 6.5|P|$$

which implies that

$$\frac{Nk(k-1) - k^2}{6.5} \leq |P|.$$

Now, since we know that $k \geq 2$ and $N \geq 4$, we have $k - 1 \geq k/2$ and $N - 2 \geq N^{1/2}$. Combining this with the previous equation yields

$$|P| \geq \frac{Nk(k-1) - k^2}{6.5} \geq \frac{(N-2)k^2}{13} \geq c_1 k^2 N^{1/2}$$

for some positive constant c_1.

In the case of (2) we have

$$33.75|P|^2(Nk)^2 \geq (|P| + Nk(k-1) - k^2)^3$$

and so

$$c_2|P|^{2/3}(Nk)^{2/3} \geq |P| + Nk(k-1) - k^2$$

for some positive constant c_2. Recall that the RHS of this inequality is $|E(G)|$, which is at least $i/2$. So we have

$$\frac{i}{2} \leq c_2|P|^{2/3}(Nk)^{2/3}.$$

Now, since each of the Nk lines in A must pass through at least k points, then there are at least Nk^2 incidences. From this it follows that

$$Nk^2 \leq i \leq c_3|P|^{2/3}(Nk)^{2/3}$$

for some positive constant c_3. Hence,

$$N^3k^6 \leq c_4|P|^2(Nk)^2$$

and so

$$|P| \geq c_5 k^2 N^{1/2}$$

for some positive constants c_4 and c_5. So in both cases, we end up with our desired result.

$\square$

Now, we can use the previous result to prove the following lemma regarding degree growth:

Lemma 7. *Given any point $p \in P_k$ with $d_k(p) = d$, there exists a positive real constant c such that*

$$d_{k+1}(p) \geq c\delta_k \left(\frac{n_k}{d}\right)^{1/2}$$

Proof. Let $p \in P_k$ with $d_k(p) = d$. By the pigeonhole principle, there exists some line ℓ through p with at least $s = \frac{n_k - 1}{d}$ points on it (excluding p). Since each of these s points has at least the minimum degree, we know that there are at least $\delta_k - 1$ lines through each point (excluding ℓ). Consider the real plane as a subset of the real projective plane in the standard way and let ℓ be the line at infinity. If we restrict our attention to only the points on ℓ and the lines through them, then we obtain a grid of $s+1$ families of parallel lines, one family for each of the points on ℓ. Each family of parallel lines contains at least $\delta_k - 1$ lines and no two families can be parallel (since they come from distinct points). We would like to restrict our attention to families of exactly $\delta_k - 1$ parallel lines. So for each family, except for the one generated by p, arbitrarily choose a subset of $\delta_k - 1$ lines and disregard all other lines in that family. Let F be the family of lines through p and choose one family R to be a set of "reference" lines. Let P_0 denote the set of points that lie at the intersection of a reference line and one of the other $s-1$ families (excluding F).

Now, during stage k, a point must be added to any intersection for which one does not already exist, in particular all points of P_0. Also, a line must be added to connect any pair of points for which one does not already exist, in particular for the pairs in the set $T = \{(p, q) \mid q \in P_0\}$. Let t denote the number of distinct lines generated by pairs in the set T. Note that any such line can pass through at most $\delta_k - 1$ points of P_0 because all the points of P_0 lie in the family R, which contains exactly $\delta_k - 1$ lines. It follows that

$$d_{k+1}(p) \geq t \geq \frac{|P_0|}{\delta_k - 1} \geq \frac{|P_0|}{\delta_k}, \tag{3}$$

with the first inequality holding because any line generated by the set T must pass through p, and hence contributes to its degree in the stage.

Now, for the moment, exclude F from our collection of families and consider all other families of lines along with the points of P_0. Suppose $s < 4$.

Hence, $n_k - 1 < 4d$ and so $n_k < 4d + 1$, i.e., $n_k \leq 4d$. It follows that

$$\left(\frac{n_k}{d}\right)^{1/2} \leq 2.$$

Note that $d_{k+1}(p) \geq \delta_k$ for all $p \in P_{k+1}$ and so $d_{k+1}(p) \geq 2c\delta_k$ holds true for $c = \frac{1}{2}$. Hence,

$$d_{k+1}(p) \geq c\delta_k \left(\frac{n_k}{d}\right)^{1/2}$$

for some positive real constant c, as desired. Now suppose that $s \geq 4$. Since we also know that $\delta_k \geq 3$, i.e., $\delta_k - 1 \geq 2$ for all $k \in \mathbb{N}$, we can apply Theorem 2 to this configuration with $F_1 = R$, $N = s$, and $k = \delta_k - 1$. It follows that

$$|P_0| \geq c_1(\delta_k - 1)^2 \left(\frac{n_k - 1}{d}\right)^{1/2}$$

for some positive constant c_1. Now, note that $\delta_k \geq 2$ and $n_k \geq 4$, which implies that $\delta_k - 1 \geq \frac{1}{2}\delta_k$ and $n_k - 1 \geq \frac{3}{4}n_k$. Combining this with the previous equation, we obtain

$$|P_0| \geq c_1 \left(\frac{\delta_k}{2}\right)^2 \left(\frac{3n_k}{4d}\right)^{1/2} \geq c_2{\delta_k}^2 \left(\frac{n_k}{d}\right)^{1/2} \tag{4}$$

for some positive constant c_2. If we combine (3) and (4) we get

$$d_{k+1}(p) \geq c_2\delta_k \left(\frac{n_k}{d}\right)^{1/2}, \tag{5}$$

as desired. This completes the proof. □

Note that $|P_0| \leq n_{k+1}$ and so

$$n_{k+1} \geq c{\delta_k}^2 \left(\frac{n_k}{d}\right)^{1/2}$$

must hold for any point $p \in P_k$ with $d_k(p) = d$ and some positive constant c. In particular, it must hold for $p \in P_k$ chosen with $d_k(p) = \delta_k$. In this case

$$n_{k+1} \geq c{\delta_k}^2 \left(\frac{n_k}{\delta_k}\right)^{1/2} = c{\delta_k}^{3/2}{n_k}^{1/2}. \tag{6}$$

Now we are able to provide an improved lower bound on the minimum degree, which will be used to improve the lower bound on n_k.

Lemma 8. *Given any $k \in \mathbb{N}$, $\epsilon \geq 0$, and any positive real constant c_1 such that $\delta_k \geq c_1 {n_k}^{\epsilon}$, there exists some positive real constant c_2 such that*

$$\delta_{k+1} \geq c_2 {n_k}^{\left(\frac{1+2\epsilon}{3}\right)}.$$

Proof. Suppose that $\delta_k \geq c_1 {n_k}^{\epsilon}$ for some $k \in \mathbb{N}$, $\epsilon \geq 0$, and positive real constant c_1. Define $\alpha \in \mathbb{R}$ by

$$\alpha = \frac{1+2\epsilon}{3}.$$

Let $p \in P_k$ with $d_k(p) = d$. There are two cases: either $d < {n_k}^{\alpha}$ or $d \geq {n_k}^{\alpha}$. If $d < {n_k}^{\alpha}$, then by Lemma 7 we have

$$d_{k+1}(p) \geq c_0 \delta_k \left(\frac{n_k}{{n_k}^{\alpha}}\right)^{1/2} = c_0 \delta_k {n_k}^{\frac{1-\alpha}{2}}$$

for some positive real constant c_0. Since $\delta_k \geq c_1 {n_k}^{\epsilon}$ and $\alpha = (1+2\epsilon)/3$, we must have

$$\begin{aligned} d_{k+1}(p) &\geq c_0 \delta_k {n_k}^{\frac{1-\alpha}{2}} \\ &\geq c_0 c_1 {n_k}^{\epsilon} {n_k}^{\frac{1-\epsilon}{3}} \\ &= c_2 {n_k}^{\left(\frac{1+2\epsilon}{3}\right)}, \end{aligned}$$

where $c_2 = c_0 c_1$. If instead $d \geq {n_k}^{\alpha}$, then obviously we have

$$d_{k+1}(p) \geq d \geq {n_k}^{\alpha} \geq c_2 {n_k}^{\left(\frac{1+2\epsilon}{3}\right)},$$

where $c_2 \leq 1$. So, in both cases, we have the conclusion that

$$d_{k+1}(p) \geq c_2 {n_k}^{\left(\frac{1+2\epsilon}{3}\right)}$$

and this will hold true for any $p \in P_k$. Since the choice of $p \in P_k$ was arbitrary, we have

$$\delta_{k+1} \geq c_2 {n_k}^{\left(\frac{1+2\epsilon}{3}\right)}, \tag{7}$$

as desired. This completes the proof. ∎

Now, we are able to obtain some numerical results from Lemma 8. Note first that

$$\delta_k \geq c_2 {n_{k-1}}^{1/3} \tag{8}$$

for some positive real constant c_2 (letting $\epsilon = 0$). Further recall that the trivial upper bound yields

$$n_{k-1} \geq (8n_k)^{1/4}. \tag{9}$$

Combining (8) and (9), we get that for all $k \in \mathbb{N}$

$$\delta_k \geq c_2 {n_{k-1}}^{1/3} \geq c_2[(8n_k)^{1/4}]^{1/3} \geq c_3 {n_k}^{1/12}$$

for some positive real constant c_3. Now, we can apply Lemma 8 with $\epsilon = \frac{1}{12}$ for any $k \in \mathbb{N}$. Since

$$\frac{1 + 2(\frac{1}{12})}{3} = \frac{7}{18},$$

we get

$$\delta_k \geq c_4 {n_{k-1}}^{7/18} \tag{10}$$

for some positive real constant c_4. Now, we combine (10) with (9), to obtain that for all $k \in \mathbb{N}$

$$\delta_k \geq c_4 {n_{k-1}}^{7/18} \geq c_4[(8n_k)^{1/4}]^{7/18} \geq c_5 {n_k}^{7/72}$$

for some positive real constant c_5. This process can be iterated and the limiting value of $\epsilon > 0$ is found by setting

$$\epsilon = \frac{1 + 2\epsilon}{12}$$

which implies that

$$\epsilon = 0.1 + o(1).$$

Now, using Lemma 8 ($\epsilon = 0.1 + o(1)$) with (6), we obtain

$$\begin{aligned} n_{k+1} &\geq c {\delta_k}^{3/2} {n_k}^{1/2} \\ &\geq c \left(c' {n_{k-1}}^{\frac{1+2(0.1+o(1))}{3}} \right)^{3/2} n_k^{1/2} \\ &\geq c'' {n_{k-1}}^{1.1+o(1)} \end{aligned} \tag{11}$$

for some positive constants c, c', and c''. Using (11) along with the trivial upper bound, we obtain the following theorem:

Theorem 9. *Given $k \in \mathbb{N}$, there exists real positive constants c_1 and c_2 such that*

$$c_1 4^{1.0488^k} \leq n_k \leq c_2 4^{4^k}. \tag{12}$$

Proof. Note first that $n_1 = 4$ and $n_2 = 7$. From repeated use of (11) we get that there exist real positive constants a_1, a_2, a_3, a_4 such that

$$a_1 4^{(1.1+o(1))^k} \leq n_{2k+1} \leq a_2 4^{4^{2k+1}}$$

and

$$a_3 7^{(1.1+o(1))^{k-1}} \leq n_{2k} \leq a_4 4^{4^{2k}}.$$

Taking square roots, it follows that there exist real positive constants c_1 and c_2 such that

$$c_1 4^{1.0488^k} \leq n_k \leq c_2 4^{4^k},$$

as desired. □

Theorem 9 shows that the growth of n_k is indeed doubly-exponential, as the easy upper bound suggests. However, a considerable gap still remains between the exponents. While we have no rigorous argument providing improvements of either bound, computational results and heuristic reasoning suggest that the actual growth rate of n_k is closer to the stated upper bound.

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