U_p-OPERATORS AND CONGRUENCES FOR SHIMURA IMAGES

MATTHEW BOYLAN

ABSTRACT. Recent works [2, 5, 13, 14] studied how Hecke operators map between subspaces of modular forms of the type $\mathcal{A}_{r,s} = \{\eta(\delta z)^r F(\delta z) : F(z) \in M_s\}$ for suitable δ. For primes $p \geq 5$, we give results on how U_p maps between subspaces of these spaces with p-integral coefficients modulo powers of p. As a corollary, we prove congruences between Shimura images of a natural family of half-integral weight eigenforms modulo p. Specifically, we classify all congruences of the type

$$\mathrm{Shim}_r(\eta(24z)^r E_s(24z)) \equiv \mathrm{Shim}_{r'}(\eta(24z)^{r'} E_{s'}(24z)) \pmod{p}$$

for $1 \leq r, r' \leq 23$ with $\gcd(rr', 6) = 1$, $5 \leq p \leq 23$, and $s, s' \in \{0, 4, 6, 8, 10, 14\}$. Congruences of this type were used in [7, 8] to extend Ramanujan's congruences for $p(n)$ modulo powers of 5, 7, and 11.

1. Statement of Results.

We let $\mathfrak{h}$ denote the complex upper half-plane, we let $z \in \mathfrak{h}$, and we let $q := e^{2\pi i z}$. As a building-block for spaces of modular forms, the Dedekind eta-function,

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.1}$$

plays a central role in this paper. See Section 2 for necessary background on modular forms. We let $1 \leq r \leq 24$ be an integer, and we define $\delta_r := \frac{24}{\gcd(24,r)}$. For square-free $t \in \mathbb{Z}$, we let D_t denote the discriminant of $\mathbb{Q}(\sqrt{t})$, and we define the Dirichlet character $\chi_t(\cdot) := \left(\frac{D_t}{\cdot}\right)$. When $N \geq 1$ is an integer, we let 1_N denote the trivial character modulo N. For all even integers s, we let M_s denote the space of modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with weight s, and we set

$$\mathcal{A}_{r,s} := \{\eta(\delta_r z)^r F(\delta_r z) : F(z) \in M_s\}. \tag{1.2}$$

Prior works [2, 5, 13, 14] used $\mathcal{S}_{r,s}$ in place of $\mathcal{A}_{r,s}$. We use $\mathcal{A}_{r,s}$ to avoid confusion with the standard notation for subspaces of cusp forms. We note that $\mathcal{A}_{r,s} = \{0\}$ when $s < 0$ or $s = 2$ since $M_s = \{0\}$ for such s. Standard facts imply that the sets $\mathcal{A}_{r,s}$ are subspaces in spaces of cusp forms $S_{r/2+s}(\Gamma_0(\delta_r^2), \psi_r)$ of weight $r/2 + s$, level δ_r^2, and nebentypus

$$\psi_r := \begin{cases} \chi_{-1}^{r/2} & r \text{ even}, \\ \chi_3 & \gcd(r, 6) = 1, \\ 1_{\delta_r} & r \in \{3, 9, 15, 21\}. \end{cases}$$

We observe, for odd r, that forms in $\mathcal{A}_{r,s}$ transform with half-integral weight and theta multiplier system.

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The works [2, 5, 13, 14] studied the effect of Hecke operators on the subspaces $\mathcal{A}_{r,s}$. When $p \geq 5$ is prime and $m \geq 0$ is an integer, we let T_{p^m} denote the Hecke operator with index p^m on the space $S_{r/2+s}(\Gamma_0(\delta_r^2), \psi_r)$. For all positive integers t, we define

$$\ell(t) := t - 24 \left\lfloor \frac{t-1}{24} \right\rfloor, \tag{1.3}$$

the least positive residue of t modulo 24. We note that when t is odd, we have $\ell(t) = t - 24 \left\lfloor \frac{t}{24} \right\rfloor$. The following theorem states to which spaces T_{p^m} maps $\mathcal{A}_{r,s}$.

Theorem 1.1. *Let $1 \leq r \leq 24$, let m and s be non-negative integers with s even, and let $p \geq 5$ be prime. Then we have*

$$T_{p^m} : \mathcal{A}_{r,s} \longrightarrow \begin{cases} \mathcal{A}_{r,s} & \text{if } m \text{ is even,} \\ \mathcal{A}_{\ell(pr),\, s + \frac{r-\ell(pr)}{2}} & \text{if } m \text{ is odd and } r \text{ is even,} \\ \{0\} & \text{if } m \text{ is odd and } r \text{ is odd.} \end{cases}$$

Garvan [5] and Yang [13] proved the theorem for odd r and $m = 2$, from which the result follows for odd r and all even m. When r and m are odd, T_{p^m} is zero on the half-integral weight spaces $S_{r/2+s}(\Gamma_0(\delta_r^2), \psi_r)$. The author and Brown proved the theorem for even r and all m in [2].

1.1. **U_p on the subspaces $\mathcal{A}_{r,s}^{(p)}$ modulo powers of p.** Let $p \geq 5$ be prime, let $1 \leq r \leq 24$, let $s \geq 0$ be even, and let $j \geq 1$. We let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at p, and we consider

$$\mathcal{A}_{r,s}^{(p)} := \{\eta(\delta_r z)^r F(\delta_r z) : F(z) \in M_s \cap \mathbb{Z}_{(p)}[\![q]\!]\}.$$

We define the U_p-operator on $\mathbb{C}[\![q]\!]$ by

$$\sum a(n) q^n \mid U_p = \sum a(pn) q^n.$$

One of our goals is to study the effect of the U_p-operator on the subspaces $\mathcal{A}_{r,s}^{(p)}$ with coefficients reduced modulo p^j. When we say that $f \in \mathcal{A}_{r,s}^{(p)} \pmod{p^j}$, we mean that there exists $g \in \mathcal{A}_{r,s}^{(p)}$ such that $f \equiv g \pmod{p^j}$ in $\mathbb{Z}_{(p)}[\![q]\!]$.

Theorem 1.2. *Let $1 \leq r \leq 24$ be an integer, let s and j be non-negative integers with s even and $j \geq 1$, and let $p \geq 5$ be prime.*

(1) *Suppose that r is odd . Then we have*

$$U_p : \mathcal{A}_{r,s}^{(p)} \longrightarrow \mathcal{A}^{(p)}_{\ell(pr),\, s + \frac{r-\ell(pr)}{2} + \frac{\phi(p^j)}{2}} \pmod{p^j}.$$

(2) *Suppose that r is even. Then we have*

$$U_p : \mathcal{A}_{r,s}^{(p)} \longrightarrow \begin{cases} \mathcal{A}^{(p)}_{\ell(pr),\, s + \frac{r-\ell(pr)}{2}} \pmod{p^j}, & j \leq s + \frac{r}{2} - 1 \\ \mathcal{A}^{(p)}_{\ell(pr),\, s + \frac{r-\ell(pr)}{2} + \phi(p^j)} \pmod{p^j}, & j > s + \frac{r}{2} - 1. \end{cases}$$

We note that forms in the image of U_p on $\mathcal{A}_{r,s}^{(p)}$ modulo powers of p have weights which tend p-adically to $r/2 + s$ as $j \to \infty$.

Remark. Instead of framing Theorem 1.2 and its proof in Section 3.1 in terms of the spaces $\mathcal{A}_{r,s}^{(p)}$, which exist at level δ_r^2 with $\delta_r = \frac{24}{\gcd(24,r)}$, one may frame the theorem and its proof in terms of spaces $\{\eta(z)^r f(z) : f(z) \in M_s\}$, which transform on $\mathrm{SL}_2(\mathbb{Z})$ with multiplier system ν_η^r, where ν_η is the eta-multiplier defined in Section 2.8 of [6] or Lemma 4 of [14]. While this approach is more direct and uniform, it may introduce some other technical issues.

We use the notion of filtration on modular forms modulo p to obtain a more precise result when $j = 1$.

Theorem 1.3. *Let $1 \le r \le 24$ be an integer, let $s \ge 0$ be an even integer, and let $p \ge 5$ be prime.*

(1) *Suppose that r is odd, and let a be the unique non-negative integer with $ap + 2 \le s + \frac{r+p}{2} \le (a+1)p + 1$. Then we have*
$$U_p : \mathcal{A}_{r,s}^{(p)} \longrightarrow \mathcal{A}^{(p)}_{\ell(pr),\, s+\frac{r-\ell(pr)}{2}+\frac{p-1}{2}-(p-1)a} \pmod{p}.$$

(2) *Suppose that r is even and that $(r,s) \neq (2,0)$. Let b be the unique non-negative integer with $bp + 2 \le s + \frac{r}{2} \le (b+1)p + 1$. Then we have*
$$U_p : \mathcal{A}_{r,s}^{(p)} \longrightarrow \mathcal{A}^{(p)}_{\ell(pr),\, s+\frac{r-\ell(pr)}{2}-(p-1)b} \pmod{p}.$$

Remark 1. We may express the integers a and b as
$$a = \left\lfloor \frac{s + \frac{r+p}{2} - 2}{p} \right\rfloor, \quad b = \left\lfloor \frac{s + \frac{r}{2} - 2}{p} \right\rfloor.$$

Remark 2. We have $s + \frac{r}{2} - 2 \le 0$ with $1 \le r \le 24$ even and $s \ge 0$ even if and only if $(r,s) = (2,0)$. In this case, we have $\mathcal{A}_{2,0} = \mathbb{C}\eta(12z)^2$ with $\eta(12z)^2 = \sum a(n)q^n \in S_1(\Gamma_0(144), \chi_{-1})$. We let $p \ge 5$ be prime. Using the fact that $\eta(12z)^2$ is a normalized newform with properties of the T_p (see (2.3)), U_p, and V_p-operators (see (2.2)), one can show that
$$\eta(12z)^2 \mid U_p \equiv \begin{cases} \eta(12z)^2(a(p) - \Delta(12z)^{\frac{p-1}{12}}) \pmod{p}, & p \equiv 1 \pmod{12}, \\ -\chi_{-1}(p)\eta(12z)^{\ell(2p)}\Delta(12z)^{\frac{2p-\ell(2p)}{24}} \pmod{p}, & p \not\equiv 1 \pmod{12} \end{cases}$$
lies in $\mathcal{A}_{\ell(2p),\, \frac{2-\ell(2p)}{2}+\phi(p)} = \mathcal{A}_{\ell(2p),\, p-\frac{\ell(2p)}{2}}$, in agreement with part (2) of Theorem 1.2 for $(r,s,j) = (2,0,1)$.

1.2. **Congruences between Shimura images on $\mathcal{A}_{r,s}^{(p)}$ modulo p.** Let $p \ge 5$ be prime. A second goal of this paper is to provide a uniform explanation for congruences between images of the Shimura Correspondence on forms in the spaces $\mathcal{A}_{r,s}^{(p)}$ modulo p. See Section 2.3 for a definition and properties of the Shimura Correspondence. Examples of congruences of this type played a central role in the work of [7, 8] on extending Ramanujan's congruences modulo powers of 5, 7, and 11 for $p(n)$, the ordinary partition function. In [8], the authors prove

$$\mathrm{Shim}_{19}(\eta(24z)^{19}) \equiv (\theta^2 E_2(z)) \otimes \chi_3 \equiv \mathrm{Shim}_{23}(\eta(24z)^{23}) \pmod{5}, \tag{1.4}$$

where $\theta = q\frac{d}{dq}$, to conclude the existence of infinitely many sub-progressions $5^j n + \beta_5(j)$ for which Ramanujan's congruence modulo 5^j can be refined to a congruence modulo 5^{j+1}. In [7], Jameson proved

$$\mathrm{Shim}_{17}(\eta(24z)^{19}) \equiv \mathrm{Shim}_{23}(\eta(24z)^{23}) \pmod{7} \tag{1.5}$$

and

$$\mathrm{Shim}_{13}(\eta(24z)^{13}E_8(24z)) \equiv \mathrm{Shim}_{23}(\eta(24z)^{23}E_8(24z)) \pmod{11}; \tag{1.6}$$

refinements to congruences for $p(n)$ modulo powers of 7 and 11 were obtained by verifying that the forms in these congruences are congruent to twists of derivatives of weight two cusp newforms with integer coefficients. These refinements, then, are phrased in terms of data associated to the corresponding elliptic curves. In this paper, we prove the following theorem on congruences for Shimura images. We recall, for all $t \geq 1$, that $\ell(t)$ is the least positive residue of t modulo 24.

Theorem 1.4. *For all values of p, s, r, $\sigma(p,s,r)$, and $\ell(pr)$ in the table below, we have*

$$\mathrm{Shim}_r(\eta(24z)^r E_s(24z)) \equiv \mathrm{Shim}_{\ell(pr)}(\eta(24z)^{\ell(pr)} E_{\sigma(p,s,r)}(24z)) \pmod{p}.$$

p	s	r	$\sigma(p,s,r)$	$\ell(pr)$
5	$0, 6$	$1, 7, 13, 19$	s	$r+4$
7	$0, 4, 8$	$1, 5, 13, 17$	s	$r+6$
11	$0, 4, 6, 8$	$1, 13$	s	$r+10$
	$0, 4, 6$	$5, 17$	s	$r+2$
13	$0, 4, 6, 8, 10, 14$	$1, 5, 7, 11$	s	$r+12$
17	$0, 4, 6, 8, 10, 14$	$1, 7$	s	$r+16$
	$0, 4, 6, 10$	$5, 11$	$s+4$	$r+8$
19	$0, 4, 6, 8, 10, 14$	$1, 5$	s	$r+18$
	$0, 4, 8$	$7, 11$	$s+6$	$r+6$
23	$0, 4, 6, 8, 10, 14$	1	s	23
	$0, 4, 6, 10$	5	$s+4$	19
	$0, 4, 8$	7	$s+6$	17
	$0, 4$	11	$s+10$	13

Remark 1. The congruences in the theorem are the complete list of congruences modulo p between images of the Shimura Correspondence on spaces $\mathcal{A}_{r,s}^{(p)}$, where $5 \leq p \leq 23$ is prime, $1 \leq r \leq 23$ has $\gcd(r,6) = 1$, and $s \in \{0,4,6,8,10,14\}$. We note that such spaces are one-dimensional. The theorem accounts for 111 congruences.

We also observe that by Theorem 1.4 of [14], the Shimura images of the forms in the theorem are newforms in $S_{r+2s-1}(\Gamma_0(6), -\chi_2(r), -\chi_3(r)) \otimes \chi_3$, the space of cusp forms of weight $r+2s-1$ on $\Gamma_0(6)$ with eigenvalues $-\chi_2(r)$ and $-\chi_3(r)$ for the Atkin-Lehner involutions

W_2 and W_3, respectively. For information on the Atkin-Lehner operators, see [1]; for information on twists of modular forms, see Chapter 7 of [6]. From this perspective, the theorem gives the complete list of congruences modulo p between newforms in the spaces $S_{r+2s-1}(\Gamma_0(6), -\chi_2(r), -\chi_3(r)) \otimes \chi_3$, where $5 \leq p \leq 23$ is prime, $1 \leq r \leq 23$ has $\gcd(r, 6) = 1$, and $s \in \{0, 4, 6, 8, 10, 14\}$.

Remark 2. We observe that the cases of $(p, s, r) \in \{(5, 0, 19), (7, 0, 23), (11, 8, 13)\}$ in the theorem give (1.4), (1.5), and (1.6) used to prove refinements of Ramanujan's congruences for $p(n)$ modulo powers or 5, 7, and 11.

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2. Background on modular forms.

For background on modular forms, one may consult [4] or [6].

2.1. **Integer weights.** We let N and k be integers with $N \geq 1$, and we let χ be a Dirichlet character modulo N. We denote the space of holomorphic modular forms on $\Gamma_0(N)$ with nebentypus χ and weight k by $M_k(\Gamma_0(N), \chi) \subseteq M_k(\Gamma_1(N))$; its subspace of cusp forms is $S_k(\Gamma_0(N), \chi)$. When $N = 1$, we write $S_k \subseteq M_k$. For $s \geq 4$ and even, we require

$$\Delta(z) := \eta(z)^{24} \in S_{12}, \quad E_s(z) := 1 - \frac{2s}{B_s} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n \in M_s, \tag{2.1}$$

where B_s is the sth Bernoulli number.

Next, we define operators on $\mathbb{C}[\![q]\!]$. For all positive integers m, we define operators U_m and V_m by

$$\sum a(n)q^n \mid U_m := \sum a(mn)q^n, \quad \sum a(n)q^n \mid V_m := \sum a(n)q^{mn}. \tag{2.2}$$

With m, N, k, and χ as above, we define the Hecke operator $T_m = T_{m,k,\chi}$ on $\mathbb{C}[\![q]\!]$ by

$$\sum a(n)q^n \mid T_m := \sum \sum_{d|\gcd(m,n)} d^{k-1}\chi(d) a\left(\frac{mn}{d^2}\right) q^n. \tag{2.3}$$

When p is prime, we find from (2.2) and (2.3) that

$$\begin{aligned} \sum a(n)q^n \mid T_p &= \sum a(n)q^n \mid U_p + p^{k-1}\chi(p) \sum a(n)q^n \mid V_p \\ &= \sum \left(a(pn) + \chi(p)p^{k-1} a\left(\frac{n}{p}\right)\right) q^n, \end{aligned} \tag{2.4}$$

with $a\left(\frac{n}{p}\right) = 0$ for $p \nmid n$. These operators map spaces of modular forms as follows:

$$\begin{aligned} &U_m : M_k(\Gamma_0(N), \chi) \longrightarrow \begin{cases} M_k(\Gamma_0(Nm), \chi) & m \nmid N, \\ M_k(\Gamma_0(N), \chi) & m \mid N; \end{cases} \\ &V_m : M_k(\Gamma_0(N), \chi) \longrightarrow M_k(\Gamma_0(Nm), \chi); \\ &T_m : M_k(\Gamma_0(N), \chi) \longrightarrow M_k(\Gamma_0(N), \chi). \end{aligned}$$

2.2. **Half-integer weights.** Next, following Shimura, we discuss modular forms of half-integral weight. For details, see for [10] and [11]. We let N and λ be integers with $N \geq 1$ and $4 \mid N$. The space of holomorphic modular forms on $\Gamma_0(N)$ with nebetypus χ and weight $\lambda + 1/2$ is $M_{\lambda+1/2}(\Gamma_0(N), \chi) \subseteq M_{\lambda+1/2}(\Gamma_1(N))$. These forms transform with the given parameters with respect to the theta-multiplier system. For all integers $m \geq 1$, let m' denote the square-free part of m. The operators U_m and V_m, as in (2.2), map spaces of half-integral weight modular forms as follows:

$$U_m : M_{\lambda+1/2}(\Gamma_0(N), \chi) \longrightarrow \begin{cases} M_{\lambda+1/2}(\Gamma_0(Nm), \chi\chi_{m'}) & m \nmid N, \\ M_{\lambda+1/2}(\Gamma_0(N), \chi\chi_{m'}) & m \mid N; \end{cases}$$

$$V_m : M_{\lambda+1/2}(\Gamma_0(N), \chi) \longrightarrow M_{\lambda+1/2}(\Gamma_0(Nm), \chi_{m'}).$$

For all integers $m \geq 1$, there are Hecke operators $T_m = T_{m,\lambda+1/2,\chi}$ which preserve the half-integral weight spaces $M_{\lambda+1/2}(\Gamma_0(N), \chi)$ and their subspaces of cusp forms. The U- and V-operators also preserve cusp conditions in both integral and half-integral weights. We note, for all $m \geq 1$, that T_m is zero when m is not a perfect square. When p is prime, we write T_{p^2} for $T_{p^2,\lambda+1/2,\chi}$ and χ^* for $\chi\chi_{-1}^{\lambda}$. We have

$$\sum a(n)q^n \mid T_{p^2} = \sum \left(a(p^2 n) + \left(\frac{n}{p}\right) \chi^*(p) p^{\lambda-1} a(n) + \chi(p)^2 p^{2\lambda-1} a\left(\frac{n}{p^2}\right) \right) q^n, \tag{2.5}$$

with $a\left(\dfrac{n}{p^2}\right) = 0$ when $p^2 \nmid n$.

2.3. **Shimura Correspondence.** We now recall basic facts on the Shimura Correspondence. We let N, $\lambda \geq 2$, and χ be as in Section 2.2, we let $f(z) = \sum a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(N), \chi)$, and we let $t \geq 1$ be square-free. For all $n \geq 1$, we define $A_t(n)$ by the formal identity

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} = L(s - \lambda + 1, \chi\chi_t\chi_{-1}^{\lambda}) \cdot \sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s},$$

where the first factor on the right is a Dirichlet L-series. It follows, for all $n \geq 1$, that

$$A_t(n) = \sum_{d \mid n} \chi\chi_{-1}^{\lambda}\chi_t \left(\frac{n}{d}\right) \left(\frac{n}{d}\right)^{\lambda-1} a(td^2). \tag{2.6}$$

We then define

$$\mathrm{Shim}_t(f) := \sum_{n=1}^{\infty} A_t(n) q^n.$$

We have the following theorem.

Theorem 2.1. *Assuming the notation above, we have* $\mathrm{Shim}_t(f) \in S_{2\lambda}(\Gamma_0(N/2), \chi^2)$. *Furthermore,* Shim_t *commutes with the actions of the appropriate Hecke operators: For all primes* p, *we have*

$$\mathrm{Shim}_t(f \mid T_{p^2,\lambda+1/2,\chi}) = \mathrm{Shim}_t(f) \mid T_{p,2\lambda,\chi^2}.$$

We recall that $f(z) \in S_{\lambda+1/2}(\Gamma_0(N), \chi)$ is an eigenform for the Hecke operator $T_{p^2,\lambda+1/2,\chi}$, where p is prime, if and only if there exists $\gamma_p \in \mathbb{C}$ such that $f(z) \mid T_{p^2,\lambda+1/2,\chi} = \gamma_p f(z)$. We now record basic, well-known facts on modular forms which are eigenforms for $T_{p^2,\lambda+1/2,\chi}$ for all primes p. These facts have short proofs. Therefore, we sketch them here.

Proposition 2.2. *Let $t, \lambda, N \geq 1$ with t square-free, $\lambda \geq 2$, and $4 \mid N$. Suppose, for all primes p, that $f(z) = \sum a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(N), \chi)$ is an eigenform for $T_{p^2,\lambda+1/2,\chi}$ with eigenvalue γ_p. Suppose also that $\mathrm{Shim}_t(f) = \sum A_t(n)q^n \in S_{2\lambda}(\Gamma_0(N/2), \chi^2)$.*

(1) *We have $\mathrm{Shim}_t(f) = 0$ if and only if $a(t) = 0$.*
(2) *Suppose that $a(t) \neq 0$. Then*
$$\frac{1}{a(t)}\mathrm{Shim}_t(f) = \sum \frac{A_t(n)}{a(t)}q^n = q + \cdots \in S_{2\lambda}(\Gamma_0(N/2), \chi^2)$$
is an eigenform for $T_{m,2\lambda,\chi^2}$ for all $m \geq 1$ with eigenvalue $\gamma_m = \dfrac{A_t(m)}{a(t)}$.
(3) *For all square-free t_1, $t_2 \geq 1$, we have*
$$a(t_1)\mathrm{Shim}_{t_2}(f) = a(t_2)\mathrm{Shim}_{t_1}(f).$$

Proof. For part (1), we observe from (2.6) that $a(t) \neq 0$ implies that $\mathrm{Shim}_t(f) = a(t)q + \cdots \neq 0$. For the reverse implication, we suppose that $a(t) = 0$. By (2.6), it suffices to show, for all $s \geq 1$, that $a(ts^2) = 0$, which one may do by induction on primes dividing s using (2.5).

For the second part, (2.4) and Theorem 2.1 imply, for all $n \geq 1$ and for all primes p, that
$$A_t(pn) + p^{2\lambda-1}\chi(p)^2 A_t\left(\frac{n}{p}\right) = \gamma_p A_t(n).$$
We set $n = 1$ and use (2.6) to obtain $A_t(p) = \gamma_p A_t(1) = \gamma_p a(t)$, from which the statement follows for p. The structure theory for Hecke operators in integer weight now implies that $\dfrac{1}{a(t)}\mathrm{Shim}_t(f)$ is an eigenform for $T_{m,2\lambda,\chi^2}$ for all $m \geq 1$ with eigenvalue $\gamma_m = \dfrac{A_t(m)}{a(t)}$.

For the third part, we first suppose that $a(t_1)a(t_2) = 0$. Then at least one of $a(t_1)$ and $a(t_2)$ is zero, say $a(t_1) = 0$. Part (1) now implies that $\mathrm{Shim}_{t_1}(f) = 0$ which gives $a(t_1)\mathrm{Shim}_{t_2}(f) = a(t_2)\mathrm{Shim}_{t_1}(f) = 0$. Next, we suppose that $a(t_1)a(t_2) \neq 0$. From part (2), it follows that
$$\frac{1}{a(t_1)}\mathrm{Shim}_{t_1}(f) = \sum \frac{A_{t_1}(n)}{a(t_1)}q^n = q + \cdots, \quad \frac{1}{a(t_2)}\mathrm{Shim}_{t_2}(f) = \sum \frac{A_{t_2}(n)}{a(t_2)}q^n = q + \cdots$$
are eigenforms for $T_{m,2\lambda,\chi^2}$ for all $m \geq 1$ with the same eigenvalues: $\dfrac{A_{t_1}(m)}{a(t_1)} = \gamma_m = \dfrac{A_{t_2}(m)}{a(t_2)}$. The result follows. □

2.4. **Modular forms modulo p.** We let $p \geq 5$ be prime. In the following proposition, we give well-known facts on the operators from Section 2.1 on formal power series with coefficients in $\mathbb{Z}_{(p)}$ reduced modulo p.

Proposition 2.3. *Let $p \geq 5$ be prime, let N and k be positive integers with $k \geq 2$, let χ be a Dirichlet character modulo N, and let $f(q)$, $g(q) \in \mathbb{Z}_{(p)}[\![q]\!]$. We have the following congruences.*

(1) $f(q) \mid U_p \equiv f(q) \mid T_{p,k,\chi} \pmod{p}$.
(2) $f(q) \mid V_p \equiv f(q)^p \pmod{p}$.
(3) $f(q)^p \mid U_p \equiv f(q) \pmod{p}$.
(4) $(f(q)^p \cdot g(q)) \mid U_p \equiv f(q) \cdot (g(q) \mid U_p) \pmod{p}$.

(5) *Let* $E_k(z)$ *be as in* (2.1). *When* $k \geq 4$ *and* $p-1 \mid k$, *we have* $E_k(z) \in \mathbb{Z}_{(p)}[\![q]\!]$ *and* $E_k(z) \equiv 1 \pmod{p}$.

Next, we let $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$, we let α be an integer or half-integer, we let $p \geq 5$ be prime, and we set $M_\alpha^{(p)}(\Gamma) := M_\alpha(\Gamma) \cap \mathbb{Z}_{(p)}[\![q]\!]$. We also suppose that j and k are non-negative integers, that $1 \leq r \leq 24$, and that $s \geq 0$ is even. We first note, from part five of the proposition, that modulo p we have

$$M_j^{(p)}(\Gamma) \subseteq M_{j+k(p-1)}^{(p)}(\Gamma), \quad \mathcal{A}_{r,s}^{(p)} \subseteq \mathcal{A}_{r,\,s+k(p-1)}^{(p)}. \tag{2.7}$$

For integers $k \geq 2$, part one of the proposition implies that $U_p : M_k^{(p)}(\Gamma) \longrightarrow M_k^{(p)}(\Gamma) \pmod{p}$. When $N = 1$ and $k > p+1$, Serre ([9], Thm. 2.3) proved that there exists an integer $\beta \geq 1$ for which $U_p : M_k^{(p)}(\Gamma) \longrightarrow M_{k-(p-1)\beta}^{(p)}(\Gamma) \pmod{p}$. Dewar [3] recently proved a precise generalization to levels $N \geq 4$, which we require:

Theorem 2.4 (Thm. 1 of [3])**.** *Suppose that* $p \geq 5$ *is prime, that* $N \neq 2, 3$ *with* $p \nmid N$, *that* $A \geq 1$, *and that* $2 \leq B \leq p+1$. *Then we have*

$$U_p : M_{Ap+B}^{(p)}(\Gamma) \longrightarrow M_{A+B}^{(p)}(\Gamma) \pmod{p}.$$

Furthermore, the map is surjective.

Remark. We may rephrase the statement of the theorem as follows: Let $k \geq 0$ be an integer. Then we have

$$U_p : M_k^{(p)}(\Gamma) \longrightarrow M_{k-(p-1)\left\lfloor \frac{k-2}{p} \right\rfloor}^{(p)}(\Gamma) \pmod{p},$$

and the map is surjective when $k \geq p+2$.

Lastly, we comment on the Shimura Correspondence on modular forms with p-integral coefficients reduced modulo p. As above, for integers or half-integers α and for primes $p \geq 5$, we set $M_\alpha^{(p)}(\Gamma_0(N), \chi) := M_\alpha^{(p)}(\Gamma_0(N), \chi) \cap \mathbb{Z}_{(p)}[\![q]\!]$. When χ is real and $t \geq 1$ is square-free, (2.6) implies that Shim_t preserves p-integrality of Fourier coefficients, and hence, congruence modulo powers of p. We have $f \in S_{\lambda+1/2}^{(p)}(\Gamma_0(N), \chi)$ implies that $\mathrm{Shim}_t(f) \in S_{2\lambda}^{(p)}(\Gamma_0(N/2))$. When j and t are positive integers with t square-free and $f,\ g \in S_{\lambda+1/2}^{(p)}(\Gamma_0(N), \chi)$ have $f(z) \equiv g(z) \pmod{p^j}$, it follows that $\mathrm{Shim}_t(f) \equiv \mathrm{Shim}_t(g) \pmod{p^j}$.

3. Proof of Theorems 1.2, 1.3, and 1.4.

We let $1 \leq r \leq 24$, $s \geq 0$ be even, and $p \geq 5$ be prime. We define

$$\mathcal{B}_{r,s} := \mathcal{A}_{r,s} \mid V_{\gcd(r,24)} = \{\eta(24z)^r F(24z) : F(z) \in M_s\}$$

and $\mathcal{B}_{r,s}^{(p)} := \mathcal{A}_{r,s}^{(p)} \mid V_{\gcd(r,24)}$. Since $p \geq 5$, the operators T_p and U_p commute with $V_{\gcd(r,24)}$. Therefore, Theorem 1.1 holds for $\mathcal{B}_{r,s}$ in place of $\mathcal{A}_{r,s}$, and it suffices to show that Theorems 1.2 and 1.3 hold for $\mathcal{B}_{r,s}^{(p)}$ in place of $\mathcal{A}_{r,s}^{(p)}$. For $j \geq 1$, Theorem 1.2 holds when $f(z) \in M_s^{(p)}$ has $\big(\eta(24z)^r f(24z)\big) \mid U_p \equiv 0 \pmod{p^j}$. Hence, we suppose that $\big(\eta(24z)^r f(24z)\big) \mid U_p \not\equiv 0 \pmod{p^j}$.

3.1. **Proof of Theorem 1.2.** We first prove part one of Theorem 1.2, in which case $1 \leq r \leq 24$ is odd. Parts (2) and (4) of Proposition 2.3 imply that

$$\begin{aligned}(3.1)\qquad \left(\eta(24z)^{p^j+r} f(24z)\right) \mid U_p &\equiv \left(\eta(24pz)^{p^{j-1}} \eta(24z)^r f(24z)\right) \mid U_p \\ &\equiv \eta(24z)^{p^{j-1}} \left(\eta(24z)^r f(24z)\right) \mid U_p \not\equiv 0 \pmod{p^j}.\end{aligned}$$

We denote the order of vanishing of the reduction modulo p^j of $g(q) \in \mathbb{Z}_{(p)}[\![q]\!]$ by $\mathrm{ord}_\infty^{(p^j)}(g)$. We now bound the order of vanishing of the q-expansion of (3.1) modulo p^j. We recall, for all integers $t \geq 1$, that $\ell(t)$ is the least positive residue of t modulo 24.

Lemma 3.1. *Let j, p, r, $s \in \mathbb{Z}$ with $j \geq 1$, $p \geq 5$ prime, $1 \leq r \leq 24$ odd, and $s \geq 0$ even, and let $f \in M_s^{(p)}$. We have*

$$ord_\infty^{(p^j)}\left((\eta(24z)^{p^j+r} f(24z)) \mid U_p\right) \geq p^{j-1} + \ell(pr).$$

Proof. Since the form is not zero modulo p^j, its order of vanishing is finite. From (1.1) and (3.1), we compute this order as

$$\begin{aligned}\mathrm{ord}_\infty^{(p^j)} \left(\eta(24z)^{p^{j-1}} (\eta(24z)^r f(24z)) \mid U_p\right) &= \mathrm{ord}_\infty^{(p^j)} \left(\eta(24z)^{p^{j-1}}\right) + \mathrm{ord}_\infty^{(p^j)} \left((\eta(24z)^r f(24z)) \mid U_p\right) \\ &= p^{j-1} + \mathrm{ord}_\infty^{(p^j)}((\eta(24z)^r f(24z)) \mid U_p).\end{aligned}$$

If we suppose that $\eta(24z)^r f(24z) = \sum_{n \equiv r \pmod{24}} a_{r,f}(n) q^n$, then we have

$$\eta(24z)^r f(24z) \mid U_p = \sum_{\substack{n \equiv r \pmod{24} \\ p \mid n}} a_{r,f}(n) q^{\frac{n}{p}} = \sum_{pn \equiv r \pmod{24}} a_{r,f}(pn) q^n = \sum_{n \equiv pr \pmod{24}} a_{r,f}(pn) q^n,$$

where the last equality holds since $p \geq 5$ is prime implies that $p^2 \equiv 1 \pmod{24}$. The conclusion follows. $\square$

Remark. Suppose that $\mathrm{ord}_\infty^{(p^j)}(f) = n_0 \geq 0$. Then we have $\eta(24z)^r f(24z) = \sum_{\substack{n \geq 24n_0+r \\ n \equiv r \pmod{24}}} a_{r,f}(n) q^n$.

In this setting, we improve the lower bound in the lemma to $p^{j-1} + \ell(pr) + n_1$, where n_1 is the least integer with $n_1 \geq \dfrac{24n_0 + r}{p}$ such that $n_1 \equiv \ell(pr) \pmod{24}$. Alternatively, this lower bound is $p^{j-1} + \ell(pr) + 24k$, where k is the least integer with $k \geq \dfrac{24n_0 + r - p\ell(pr)}{24p}$.

The next lemma identifies the space of type (1.2) in which the form in (3.1) lies modulo p^j.

Lemma 3.2. *Let j, p, r, s, and f be as in Lemma 3.1. Modulo p^j, we have*

$$\left(\eta(24z)^{p^j+r} f(24z)\right) \mid U_p \in \begin{cases} \mathcal{B}^{(p)}_{\ell(pr)+p-24\left\lfloor \frac{\ell(pr)+p-1}{24}\right\rfloor,\ s+\frac{r-\ell(pr)}{2}+\frac{p^j-p}{2}+12\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor}, & j \text{ even,} \\ \mathcal{B}^{(p)}_{\ell(pr)+1,\ s+\frac{r-\ell(pr)}{2}+\frac{p^j-1}{2}}, & j \text{ odd.}\end{cases}$$

The proof of Lemma 3.2 requires the following proposition.

Proposition 3.3. *Let j, p and r be as in Lemma 3.1. We have*

$$\ell(r+p^j) = \begin{cases} r+1, & j \text{ even}, \\ \ell(r+p) = r+p-24\left\lfloor \frac{r+p-1}{24} \right\rfloor, & j \text{ odd}. \end{cases}$$

Proof. Since $p \geq 5$ is prime, we have $p^j \equiv \begin{cases} 1 \pmod{24}, & j \text{ even}, \\ p \pmod{24}, & j \text{ odd}. \end{cases}$ Recalling from (1.3) that $\ell(t)$ is the least positive residue of t modulo 24, the statement follows. $\square$

We turn to the proof of Lemma 3.2.

Proof of Lemma 3.2. Noting (2.1) and that $p^j + r - \ell(p^j + r) \equiv 0 \pmod{24}$, we have

$$\begin{aligned} \eta(24z)^{p^j+r} f(24z) &= \eta(24z)^{\ell(p^j+r)} \eta(24z)^{p^j+r-\ell(p^j+r)} f(24z) \\ &= \eta(24z)^{\ell(p^j+r)} \Delta(24z)^{\frac{p^j+r-\ell(p^j+r)}{24}} f(24z) \in \mathcal{B}^{(p)}_{\ell(p^j+r),\ s+\frac{p^j+r-\ell(p^j+r)}{2}}. \end{aligned}$$

Using Proposition 3.3, we obtain

$$\eta(24z)^{p^j+r} f(24z) = \begin{cases} \eta(24z)^{r+1} \Delta(24z)^{\frac{p^j-1}{24}} f(24z) \in \mathcal{B}^{(p)}_{r+1,\ s+\frac{p^j-1}{2}}, & j \text{ even}, \\ \eta(24z)^{\ell(r+p)} \Delta(24z)^{\frac{p^j+r-\ell(r+p)}{24}} f(24z) \in \mathcal{B}^{(p)}_{\ell(r+p),\ s+\frac{p^j+r-\ell(r+p)}{2}}, & j \text{ odd}. \end{cases}$$

Since r is odd and $p \geq 5$, these subspaces consist of forms of weight $s + \frac{p^j+r}{2} \in \mathbb{Z}$, and the power of the eta-function which occurs as a factor of forms in these subspaces is even. Furthermore, $r \geq 1$, $s \geq 0$, and $p \geq 5$ imply that $s + \frac{p^j+r}{2} - 1 \geq \frac{5^j-1}{2} \geq j$ for all $j \geq 1$. It follows from (2.4) that U_p and T_p agree modulo p^j on these subspaces. Hence, to determine where U_p modulo p^j maps $\mathcal{B}^{(p)}_{r,s}$, we apply Theorem 1.1 with U_p in place of T_p and $\mathcal{B}_{r,s}$ in place of $\mathcal{A}_{r,s}$. Recalling from (1.3) that $\ell(t)$ is the least positive residue of t modulo 24, we observe that

$$\ell(p(r+1)) = \ell(pr) + p - 24\left\lfloor \frac{\ell(pr)+p-1}{24} \right\rfloor \tag{3.2}$$

and, since p and r are odd with $p \geq 5$, that

$$\ell(p\ell(r+p)) = \ell(pr) + 1. \tag{3.3}$$

Hence, when j is even, Theorem 1.1 and (3.2) give U_p modulo p^j mapping $\mathcal{B}^{(p)}_{r+1,\ s+\frac{p^j-1}{2}}$ to

$$\mathcal{B}^{(p)}_{\ell(p(r+1)),\ s+\frac{p^j-1}{2}+\frac{r+1-\ell(p(r+1))}{2}} = \mathcal{B}^{(p)}_{\ell(pr)+p-24\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor,\ s+\frac{r-\ell(pr)}{2}+\frac{p^j-p}{2}+12\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor}.$$

When j is odd, Theorem 1.1 and (3.3) give U_p modulo p^j mapping $\mathcal{B}^{(p)}_{\ell(p+r),\ s+\frac{p^j+r-\ell(r+p)}{2}}$ to

$$\mathcal{B}^{(p)}_{\ell(p(\ell(r+p))),\ s+\frac{p^j+r-\ell(r+p)}{2}+\frac{\ell(r+p)-\ell(p(\ell(r+p)))}{2}} = \mathcal{B}^{(p)}_{\ell(pr)+1,\ s+\frac{r-\ell(pr)}{2}+\frac{p^j-1}{2}}.$$

This proves Lemma 3.2. $\square$

We now use Lemmas 3.1 and 3.2 to prove part one of Theorem 1.2. We recall that $1 \le r \le 24$ is odd in part one of the theorem. A consequence of Lemma 3.2 is that there exists $g(z) \in \mathbb{Z}_{(p)}[\![q]\!]$, depending on j, p, r, and f, with

$$g(z) \in \begin{cases} M^{(p)}_{s+\frac{r-\ell(pr)}{2}+\frac{p^j-p}{2}+12\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor}, & j \text{ even},\\ M^{(p)}_{s+\frac{r-\ell(pr)}{2}+\frac{p^j-1}{2}}, & j \text{ odd}\end{cases} \tag{3.4}$$

and

$$\left(\eta(24z)^{p^j+r}f(24z)\right) \mid U_p \equiv \begin{cases} \eta(24z)^{\ell(pr)+p-24\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor} g(24z) \pmod{p^j}, & j \text{ even},\\ \eta(24z)^{\ell(pr)+1} g(24z) \pmod{p^j}, & j \text{ odd}.\end{cases} \tag{3.5}$$

Lemma 3.1 implies that

$$\begin{aligned}\operatorname{ord}_\infty^{(p^j)}\left(\left(\eta(24z)^{p^j+r}f(24z)\right) \mid U_p\right) &= \begin{cases} \ell(pr) + 24\operatorname{ord}_\infty^{(p^j)}(g(z)) + p - 24\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor, & j \text{ even},\\ \ell(pr) + 24\operatorname{ord}_\infty^{(p^j)}(g(z)) + 1, & j \text{ odd}\end{cases}\\ &\ge p^{j-1} + \ell(pr).\end{aligned}$$

We conclude that $g(z)$ has q-expansion modulo p^j with order of vanishing

$$\operatorname{ord}_\infty^{(p^j)}\left(g(z)\right) \ge \begin{cases} \frac{p^{j-1}-p}{24} + \left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor, & j \text{ even},\\ \frac{p^{j-1}-1}{24}, & j \text{ odd}.\end{cases} \tag{3.6}$$

We note that the expressions on the right side of the inequalities are integers since $p \ge 5$ is prime. We next define

$$h(z) := \begin{cases} g(z)\,\Delta(z)^{-\left(\frac{p^{j-1}-p}{24}+\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor\right)}, & j \text{ even},\\ g(z)\,\Delta(z)^{-\left(\frac{p^{j-1}-1}{24}\right)}, & j \text{ odd}.\end{cases}$$

We use (3.4) and (3.6) to deduce, for both even and odd j, that $h(z) \in M^{(p)}_{s+\frac{r+\ell(pr)}{2}+\frac{\phi(p^j)}{2}}$ is a holomorphic modular form, where ϕ is Euler's phi-function. We may therefore rewrite (3.5) as

$$\left(\eta(24z)^{p^j+r}f(24z)\right) \mid U_p \equiv \begin{cases} \eta(24z)^{\ell(pr)+p-24\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor}\Delta(24z)^{\frac{p^{j-1}-p}{24}+\left\lfloor\frac{\ell(pr)+p-1}{24}\right\rfloor}h(24z), & j \text{ even},\\ \eta(24z)^{\ell(pr)+1}\Delta(24z)^{\frac{p^{j-1}-1}{24}}h(24z), & j \text{ odd}.\end{cases}$$

Hence, for both even and odd j, we find that

$$\left(\eta(24z)^{p^j+r}f(24z)\right) \mid U_p \equiv \eta(24z)^{\ell(pr)+p^{j-1}}h(24z) \pmod{p^j}. \tag{3.7}$$

From (3.1) and (3.7), we now have

$$\eta(24z)^{p^{j-1}}\left(\eta(24z)^r f(24z)\right) \mid U_p \equiv \left(\eta(24z)^{p^j+r}f(24z)\right) \mid U_p \equiv \eta(24z)^{\ell(pr)+p^{j-1}}h(24z) \pmod{p^j}.$$

Part one of Theorem 1.2 follows for all $j \ge 1$.

When r is even and $s + \frac{r}{2} - 1 \ge j$, we find that U_p and T_p agree modulo p^j on $\mathcal{A}^{(p)}_{r,s}$; we then apply Theorem 1.1 with U_p in place of T_p and $\mathcal{B}^{(p)}_{r,s}$ in place of $\mathcal{A}^{(p)}_{r,s}$ to prove part two of the theorem for such j. To conclude the proof of Theorem 1.2, we explain how to adapt the argument for part one to prove part two when $j > s + \frac{r}{2} - 1$. We suppose that $f(z) \in M_s^{(p)}$ has $(\eta(24z)^r f(24z)) \mid U_p \not\equiv 0 \pmod{p^j}$. In analogy with (3.1), we first note that

$$(\eta(24z)^{2p^j+r} f(24z)) \mid U_p \equiv \eta(24z)^{2p^{j-1}} (\eta(24z)^r f(24z)) \mid U_p \pmod{p^j}.$$

Since r is even and $p \geq 5$ is odd, the exponent $2p^j + r$ is even, which allows us to use Theorem 1.1, to study U_p modulo p^j on the relevant spaces, as in the proof of part one of the theorem. As the details of the argument are not significantly different, we omit them in the interest of brevity.

3.2. **Proof of Theorem 1.3.** Let $1 \leq r \leq 24$, let $s \geq 0$ be even, let $p \geq 5$ be prime, and let $\Gamma = \Gamma_0(576)$. We first prove Theorem 1.3 for even r. When $(r, s) \neq (2, 0)$, the operators U_p and T_p agree modulo p on $\mathcal{B}_{r,s}^{(p)}$. Theorem 1.1 implies that U_p maps $\mathcal{B}_{r,s}^{(p)} \subseteq M_{s+\frac{r}{2}}^{(p)}(\Gamma)$ to $\mathcal{B}_{\ell(pr),\, s+\frac{r-\ell(pr)}{2}}^{(p)} \subseteq M_{s+\frac{r}{2}}^{(p)}(\Gamma)$ modulo p, while Theorem 2.4 implies that U_p maps $M_{s+\frac{r}{2}}^{(p)}(\Gamma) \to M_{s+\frac{r}{2}-(p-1)b}^{(p)}(\Gamma)$ modulo p, where $b = \left\lfloor \frac{s+\frac{r}{2}-2}{p} \right\rfloor$. We use (2.7) to conclude that the image of U_p on $\mathcal{B}_{r,s}^{(p)}$ modulo p lies in

$$\mathcal{B}_{\ell(pr),s+\frac{r-\ell(pr)}{2}}^{(p)} \cap M_{s+\frac{r}{2}-(p-1)b}^{(p)}(\Gamma) = \mathcal{B}_{\ell(pr),s+\frac{r-\ell(pr)}{2}-(p-1)b}^{(p)}.$$

When r is odd, we modify the proof of part one of Theorem 1.2 in order to apply Theorem 2.4. With $f(z) \in M_s^{(p)}$, we recall the $j = 1$ case of (3.1):

$$(\eta(24z)^{p+r} f(24z)) \mid U_p \equiv \eta(24z) \left(\eta(24z)^r f(24z)\right) \mid U_p \pmod{p}. \tag{3.8}$$

Since r and p are odd imply that $\frac{r+p}{2}$ is an integer, we observe that $\eta(24z)^{p+r} f(24z) \in M_{s+\frac{p+r}{2}}^{(p)}(\Gamma)$ has integer weight. Therefore, Theorem 2.4 implies that (3.8) lies in $M_{s+\frac{p+r}{2}-(p-1)a}^{(p)}(\Gamma)$ modulo p, where $a = \left\lfloor \frac{s+\frac{p+r}{2}-2}{p} \right\rfloor$. On the other hand, by the $j = 1$ case of Lemma 3.2, the form (3.8) lies in $\mathcal{B}_{\ell(pr)+1,\, s+\frac{r-\ell(pr)}{2}+\frac{p-1}{2}}^{(p)} \subseteq M_{s+\frac{p+r}{2}}^{(p)}(\Gamma)$ modulo p. It follows from (2.7) that (3.8) lies in

$$\mathcal{B}_{\ell(pr)+1,\, s+\frac{r-\ell(pr)}{2}+\frac{p-1}{2}}^{(p)} \cap M_{s+\frac{p+r}{2}-(p-1)a}^{(p)}(\Gamma) = \mathcal{B}_{\ell(pr)+1,\, s+\frac{r-\ell(pr)}{2}+\frac{p-1}{2}-(p-1)a}^{(p)}.$$

We conclude as in the proof of part one of Theorem 1.2. We now have $g(z) \in M_{s+\frac{r-\ell(pr)}{2}+\frac{p-1}{2}-(p-1)a}^{(p)}$ such that

$$(\eta(24z)^{p+r} f(24z)) \mid U_p \equiv \eta(24z)^{\ell(pr)+1} g(24z) \pmod{p}.$$

We compare with (3.8) to obtain

$$\eta(24z)(\eta(24z)^r f(24z)) \mid U_p \equiv \eta(24z)^{\ell(pr)+1} g(24z) \pmod{p},$$

which gives the result.

3.3. **Proof of Theorem 1.4.** We let N and λ be positive integers with $\lambda \geq 2$ and $4 \mid N$, we let χ be a Dirichlet character modulo N, and we let $f(z) = \sum a(n) q^n \in S_{\lambda+1/2}(\Gamma_0(N), \chi)$. We require a lemma and a corollary on the interplay between the Shimura Correspondence and U-operator on the form f.

Lemma 3.4. *Let i and $j \geq 1$ with i square-free and $j \mid i$, and let f be as in the preceding paragraph. Then we have*

$$\mathrm{Shim}_i(f) = \mathrm{Shim}_{i/j}(f \mid U_j).$$

Proof. We note that j and i/j are square-free. For all $n \geq 1$, we define $b_j(n)$ by

$$\sum b_j(n)q^n = f \mid U_j \in S_{\lambda+\frac{1}{2}}(\Gamma_0(Nj), \chi\chi_j).$$

Therefore, from (2.2) we have $b_j(n) = a(jn)$. We also find that

$$\chi_j\chi_{i/j} = \begin{cases} \chi_i\, 1_2, & i \equiv 1 \pmod 4 \text{ and } j \equiv 3 \pmod 4, \\ \chi_i, & \text{otherwise.} \end{cases}$$

Since $4 \mid N$ and χ is a Dirichlet character with modulus N, it follows that $\chi\chi_j\chi_{-1}^{\lambda}\chi_{i/j} = \chi\chi_{-1}^{\lambda}\chi_i$. We now use (2.6) to compute

$$\begin{aligned} \mathrm{Shim}_{i/j}(f \mid U_j) &= \sum \left(\sum_{d \mid n} \chi\chi_j\chi_{-1}^{\lambda}\chi_{i/j}(d)d^{\lambda-1}b_j\left(\frac{i}{j} \cdot \frac{n^2}{d^2}\right)\right) q^n \\ &= \sum \left(\sum_{d \mid n} \chi\chi_{-1}^{\lambda}\chi_i(d)d^{\lambda-1}a\left(i\,\frac{n^2}{d^2}\right)\right) q^n = \mathrm{Shim}_i(f). \end{aligned}$$

□

The following corollary is an application of Proposition 2.2 and the lemma.

Corollary 3.5. *Suppose, for all primes ℓ, that $f(z)$ is an eigenform for the Hecke operator $T_{\ell^2,\lambda+1/2,\chi}$. Let t_1, t_2, and $m \geq 1$ be square-free with $\gcd(m, t_2) = 1$. Then we have*

$$a(mt_2)\mathrm{Shim}_{t_1}(f) = a(t_1)\mathrm{Shim}_{t_2}(f \mid U_m).$$

Remark. Let $p \geq 5$ be prime, and let $j \geq 1$. When χ is real and $f(z) \in S^{(p)}_{\lambda+1/2}(\Gamma_0(N), \chi)$, the statement of the corollary holds modulo p^j.

Proof. We note that since m and t_2 are square-free and coprime, we have mt_2 square-free. We find that

$$a(mt_2)\mathrm{Shim}_{t_1}(f) = a(t_1)\mathrm{Shim}_{mt_2}(f) = a(t_1)\mathrm{Shim}_{t_2}(f \mid U_m),$$

where the first equality follows from part (3) of Proposition 2.2, and the second follows from the lemma with $i = mt_2$ and $j = t_2$. □

We now embark on the proof of Theorem 1.4. We let p, s, and r be as in the theorem, and we set

$$f_{r,s}(z) = \eta(24z)^r E_s(24z) = \sum a_{r,s}(n)q^n \in \mathcal{A}^{(p)}_{r,s}.$$

Since $\mathcal{A}^{(p)}_{r,s}$ is one-dimensional for such p, s, and r, Theorem 1.1 implies that the forms $f_{r,s}(z)$ are eigenforms for T_{ℓ^2} for all primes $\ell \geq 5$. Furthermore, since these forms have support on indices relatively prime to 6, formula (2.5) implies that they are eigenforms for T_4 and T_9 with eigenvalues both equal to zero. Therefore, we may apply Corollary 3.5 to them with $t_1 = r$, $t_2 = \ell(pr)$, and $m = p$ to obtain

$$a_{r,s}(p\ell(pr))\mathrm{Shim}_r(\eta(24z)^r E_s(24z)) = \mathrm{Shim}_{\ell(pr)}((\eta(24z)^r E_s(24z)) \mid U_p). \tag{3.9}$$

We suppose that $v_p((\eta(24z)^r E_s(24z)) \mid U_p) = v \geq 0$. We have $a_{r,s}(p\ell(pr)) \equiv 0 \pmod{p^v}$ from (3.9). Dividing by p^v in (3.9) yields

$$\frac{a_{r,s}(p\ell(pr))}{p^v}\mathrm{Shim}_r(\eta(24z)^r E_s(24z)) = \mathrm{Shim}_{\ell(pr)}\left(\frac{(\eta(24z)^r E_s(24z)) \mid U_p}{p^v}\right) \in \mathbb{Z}_{(p)}[\![q]\!].$$

Therefore, to complete the proof of Theorem 1.4, it suffices to show, in the notation of the theorem, that

$$\frac{(\eta(24z)^r E_s(24z)) \mid U_p}{p^v} \equiv \frac{a_{r,s}(p\ell(pr))}{p^v}\eta(24z)^{\ell(pr)}E_{\sigma(p,s,r)}(24z) \pmod{p}.$$

When $v = 0$, part one of Theorem 1.3 gives $(\eta(24z)^r E_s(24z)) \mid U_p \in \mathcal{B}^{(p)}_{\ell(pr),\sigma(p,s,r)} \pmod{p}$. In all of the cases we consider, this space is one-dimensional spanned by $\eta(24z)^{\ell(pr)}E_{\sigma(p,s,r)}(24z)$. The result follows.

When $v > 0$, part one of Theorem 1.2 implies that $(\eta(24z)^r E_s(24z)) \mid U_p \not\equiv 0 \pmod{p^{v+1}}$ lies in $\mathcal{B}^{(p)}_{\ell(pr),s+\frac{r-\ell(pr)}{2}+\frac{\phi(p^{v+1})}{2}}$ modulo p^{v+1}. Hence, there exists $c(z) \in M^{(p)}_{s+\frac{r-\ell(pr)}{2}+\frac{\phi(p^{v+1})}{2}}$ with

$$(\eta(24z)^r E_s(24z)) \mid U_p \equiv a_{r,s}(p\ell(pr))\eta(24z)^{\ell(pr)}c(24z) \pmod{p^{v+1}},$$

which gives

$$\frac{(\eta(24z)^r E_s(24z)) \mid U_p}{p^v\eta(24z)^{\ell(pr)}} \equiv \frac{a_{r,s}(p\ell(pr))}{p^v}c(24z) \pmod{p} \tag{3.10}$$

in $M^{(p)}_{s+\frac{r-\ell(pr)}{2}+\frac{\phi(p^{v+1})}{2}}(\Gamma_0(24))$. Since the form on the right side of the congruence arises as the image of V_{24} on a form on $\mathrm{SL}_2(\mathbb{Z})$, we apply U_{24} to see that (3.10) is equivalent to

$$\left(\frac{(\eta(24z)^r E_s(24z)) \mid U_p}{p^v\eta(24z)^{\ell(pr)}}\right) \mid U_{24} \equiv \frac{a_{r,s}(p\ell(pr))}{p^v}c(z) \pmod{p}$$

in $M^{(p)}_{s+\frac{r-\ell(pr)}{2}+\frac{\phi(p^{v+1})}{2}}$. A theorem of Sturm [12] states that two forms in $M^{(p)}_k(\Gamma_0(N))$ agree modulo p if and only if their coefficients agree modulo p up to index $\frac{kN}{12}\prod\left(1+\frac{1}{\ell}\right)$, where the product is over primes $\ell \mid N$. The spaces $M^{(p)}_{s+\frac{r-\ell(pr)}{2}+\frac{\phi(p^{v+1})}{2}}$ have Sturm bound $2s + r - \ell(pr) + \phi(p^{v+1})$. We conclude the theorem for (p, s, r) with $v > 0$ on a case-by-case basis using the Sturm bound to show in each case that $c(z) \equiv E_{\sigma(p,s,r)}(z) \pmod{p}$.

Example. We illustrate the general case when $v > 0$ with a typical example: $(p, s, r) = (19, 8, 11)$. We observe that $\sigma(19, 8, 11) = 14$. Part one of Theorem 1.3 gives $(\eta(24z)^{11}E_8(24z)) \mid U_{19} \in \mathcal{B}^{(19)}_{17,-4} \pmod{19}$, which implies that $(\eta(24z)^{11}E_8(24z)) \mid U_{19} \equiv 0 \pmod{19}$. The argument in the preceding paragraph gives $c(z) \in M^{(19)}_{176}$ with

$$\left(\frac{(\eta(24z)^{11}E_8(24z)) \mid U_{19}}{19\eta(24z)^{17}}\right) \mid U_{24} \equiv 5c(z) \pmod{19}.$$

To prove that $c(z) \equiv E_{14}(z) \pmod{19}$, we note that $E_{14}(z) \equiv E_{14}(z)E_{18}(z)^9 \pmod{19}$ in $M^{(19)}_{176}$, and we check that the first $\left\lfloor \frac{176}{12} \right\rfloor$ coefficients agree modulo 19.

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Department of Mathematics, University of South Carolina, Columbia, SC 29208

E-mail address: `boylan@math.sc.edu`