Abstract

In recent work [9], Guerzhoy, Kent, and Ono proved a result in the theory of harmonic weak Maass forms on the p-adic coupling of mock modular forms and their shadows. As an application, they construct a sequence of weakly holomorphic modular forms whose p-adic limit gives the modular parametrization of the cubic arithmetic-geometric mean (AGM). In this paper, we construct odd weight weakly holomorphic forms with non-trivial character to prove an analogous p-adic limit formula for the modular parametrization of Gauss' quadratic AGM.

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Quadratic AGM and p-adic Limits Arising from Modular Forms

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1. *Introduction and statement of results.*

Harmonic weak Maass forms are currently the subject of much interest. They play a vital role in the study of mock theta functions, partition statistics, and related q-series. They also have applications to the study of number-theoretic objects, such as modular L-functions, singular moduli, and Borcherds products; see for example [13] and [15] and the references therein. In this paper, we apply the theory of harmonic weak Maass forms to the study of modular forms related to Gauss' quadratic arithmetic-geometric mean (AGM).

To make this precise, we set some notation. Let $k \geq 2$ and $N \geq 1$ be integers, let χ be a Dirichlet character modulo N, and let $H_{2-k}(\Gamma_0(N), \chi)$ denote the space of harmonic weak Maass forms of weight $2-k$ on the congruence subgroup $\Gamma_0(N)$ with character χ; see Section 2 for details and definitions. The differential operator

$$\xi_{2-k} := 2iy^{2-k} \cdot \overline{\frac{\partial}{\partial \overline{z}}} \tag{1.1}$$

is central to the study of the spaces $H_{2-k}(\Gamma_0(N), \chi)$. If we denote the space of cusp forms of weight k on $\Gamma_0(N)$ with character χ by $S_k(\Gamma_0(N), \chi)$, then we have

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N), \chi) \to S_k(\Gamma_0(N), \overline{\chi}).$$

Moreover, a harmonic weak Maass form $f(z) \in H_{2-k}(\Gamma_0(N), \chi)$ canonically decomposes as the sum of a holomorphic part, $f^+(z)$, and a non-holomorphic part, $f^-(z)$. If $q := e^{2\pi i z}$, then $f^+(z)$ is expressible as a q-series called a mock modular form. We refer to the cusp form $\xi_{2-k}(f)$ as the shadow of $f^+(z)$.

In [9], Guerzhoy, Kent, and Ono demonstrate how to uniformly adjust mock modular forms to obtain q-series with algebraic coefficients called regularized mock modular forms. The main result in this work then explicitly links regularized mock modular forms to their shadows via a p-adic limit formula; see Section 3. As an application, the authors express the hypergeometric function which parametrizes the cubic arithmetic-geometric mean as a p-adic limit of modular forms.

In view of this work, we consider the corresponding situation for Gauss' quadratic AGM. For positive real numbers a and b with $b \leq a$, we denote by $\mathrm{M}(a, b)$ the arithmetic-geometric mean of a and b. The function $\mathrm{M}(a, b)$ is the coincidental limit of two sequences $\{a_n\}$ and $\{b_n\}$. Let $a_0 := a$, $b_0 := b$, and for all $n \geq 1$, define

$$a_n := \frac{a_{n-1} + b_{n-1}}{2}, \quad b_n := \sqrt{a_{n-1} b_{n-1}}. \tag{1.2}$$

These sequences rapidly converge to the common limit

$$\mathrm{M}(a, b) := \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

For more information on the AGM, see [8]. Let $(a)_0 := 1$ and for all integers $n \geq 1$, let

$$(a)_n := a(a+1)\cdots(a+n-1).$$

For $|x| < 1$, one defines the Gaussian hypergeometric series by

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n.$$

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References for the special functions in this paper are [1] and [2]. Gauss proved for $0 < x < 1$ that

$$\frac{1}{\mathrm{M}(1,x)} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x^2\right).$$

It is also possible to extend the definition of $\mathrm{M}(a,b)$ to non-zero $a, b \in \mathbb{C}$, taking care with the choice of square root; see for example [8]. We now consider a modular parametrization of complex $\mathrm{M}(1,x)$. Let $\eta(z)$ be the Dedekind eta-function. We require the following theta functions:

$$\begin{aligned}
\theta_2(z) &:= 2 \cdot \frac{\eta(4z)^2}{\eta(2z)} = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}, \\
\theta_3(z) &:= \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \\
\theta_4(z) &:= \frac{\eta(z)^2}{\eta(2z)} = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}.
\end{aligned} \tag{1.3}$$

We also require the auxiliary forms

$$a(z) := \theta_3(z)^2, \quad b(z) := \theta_4(z)^2, \quad c(z) := \theta_2(z)^2. \tag{1.4}$$

It is well-known that these forms satisfy

$$a(2z) = \frac{a(z)+b(z)}{2}, \tag{1.5}$$

$$b(2z) = \sqrt{a(z)b(z)}, \tag{1.6}$$

$$c(2z) = \frac{a(z)-b(z)}{2}. \tag{1.7}$$

Note that formulas (1.5) and (1.6) parametrize the iteration of the sequences $\{a_n\}$ and $\{b_n\}$ as in (1.2) with $a_0(z) = a(z)$, $b_0(z) = b(z)$, $a_n(z) = a(2^n z)$, and $b_n(z) = b(2^n z)$. Next, set

$$x(z) := \frac{b(z)^2}{a(z)^2}. \tag{1.8}$$

The Gaussian hypergeometric function satisfies the modular parametrization

$$F(x(z)) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-x(z)\right) = a(z) = 1 + 4q + 4q^2 + 4q^4 + 8q^5 + 4q^8 + \cdots. \tag{1.9}$$

Applying Pfaff's transformation [2, Theorem 2.2.5] to (1.9) yields

$$F\left(\frac{1}{x(z)}\right) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\frac{1}{x(z)}\right) = b(z) = 1 - 4q + 4q^2 + 4q^4 - 8q^5 + 4q^8 + \cdots. \tag{1.10}$$

Furthermore, if we define

$$G(x(z)) := F(x(z)) - F\left(\frac{1}{x(z)}\right), \tag{1.11}$$

then it follows by (1.7), (1.9), and (1.10) that

$$(1/2) \cdot G(x(z/2)) = c(z). \tag{1.12}$$

Our result expresses the q-series $a(2z)$ and $b(2z)$ as p-adic limits of sequences of q-series constructed using harmonic weak Maass forms for certain primes $p \equiv 3 \pmod 4$. For this purpose, we require

$$\Omega(z) = \sum_{n=-1}^{\infty} C(n)q^n := \frac{F(x(4z))^2 \cdot F(1/x(2z)) \cdot G(x(z/2))^2}{G(x(z))^2} = q^{-1} - 2q^3 - 13q^7 + \cdots. \tag{1.13}$$

Alternatively, we note that (1.3), (1.4), (1.9), (1.10), and (1.12) imply that

$$\Omega(z) = \frac{\eta(4z)^2\eta(8z)^{12}}{\eta(16z)^8}. \tag{1.14}$$

We now state our result. It is the analogue for the quadratic AGM of Theorem 1.3 of [9].

Theorem 1.1. *Let $x(z)$ be as in (1.8), $G(x(z))$ as in (1.11), and $\Omega(z)$ and $C(n)$ as in (1.13). Suppose that $p \equiv 3 \pmod 4$ is prime with $p^2 \nmid C(p)$. Then, as p-adic limits, we have*

$$
{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{1}{x(2z)}\right) = \left(\frac{8}{G(x(z/2))}\right)^2 \cdot \lim_{w\to\infty} \frac{\Omega(z) \mid U(p^{2w+1})}{C(p^{2w+1})},
$$

$$
{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-x(2z)\right) = \frac{1}{\sqrt{x(2z)}} \cdot \left(\frac{8}{G(x(z/2))}\right)^2 \cdot \lim_{w\to\infty} \frac{\Omega(z) \mid U(p^{2w+1})}{C(p^{2w+1})}.
$$

Remark 1. From (1.13) and (1.14) we see that $C(n)$ has support in the progression $n \equiv 3 \pmod 4$, which explains why the theorem does not apply to primes $p \equiv 1 \pmod 4$ nor to even powers of primes $p \equiv 3 \pmod 4$.

Remark 2. The hypothesis $p^2 \nmid C(p)$ implies that the p-adic limits in the theorem exist and are non-zero. Z. Kent has confirmed this hypothesis for $p \leq 31,500$ using PARI/GP.

We elucidate the p-adic convergence in the theorem with an example. We see by (1.13) that the prime $p = 3$ satisfies the hypotheses. Consider for integers $w \geq 0$ the function

$$
\Omega(2w+1;z) := \left(\frac{8}{G(x(z/2))}\right)^2 \cdot \frac{\Omega(z) \mid U(3^{2w+1})}{C(3^{2w+1})}.
$$

Computations reveal that

$$
\Omega(1;z) = 1 - 4q^2 - \frac{55}{2}q^4 + 126q^6 - 554q^8 + 1720q^{10} + \cdots \equiv F\left(\frac{1}{x(2z)}\right) \pmod{3^2},
$$

$$
\Omega(3;z) = 1 - 4q^2 - \frac{16015}{26}q^4 + \frac{32238}{13}q^6 - \frac{58879}{2}q^8 + \cdots \equiv F\left(\frac{1}{x(2z)}\right) \pmod{3^4},
$$

$$
\Omega(5;z) = 1 - 4q^2 - \frac{1837035089515}{134246}q^4 + \frac{3674071252998}{67123}q^6 + \cdots \equiv F\left(\frac{1}{x(2z)}\right) \pmod{3^6}.
$$

In particular, the 3-adic convergence of the sequence $\Omega(2w+1;z)$ to

$$
F\left(\frac{1}{x(2z)}\right) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\frac{1}{x(2z)}\right) = b(2z) = 1 - 4q^2 + 4q^4 + 4q^8 - 8q^{10} + 4q^{16} + \cdots
$$

begins to emerge.

We structure the paper as follows. In Section 2, we give necessary facts on Poincaré series, harmonic weak Maass forms, and differential operators. In Section 3, we state results from [9] that we require, and we give a lemma on the existence of weakly holomorphic modular forms with prescribed principal parts. In Section 4, we prove Theorem 1.1.

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To be entered later.

2. *Poincaré series and harmonic weak Maass forms.*

We first define Poincaré series relevant to the proof of Theorem 1.1; references for holomorphic Poincaré series include [11,14]; references for non-holomorphic Poincaré series include [3,4,10]. Let $k > 2$, $N \geq 1$, and t be integers, let χ be a Dirichlet character modulo N, and set $e(z) := \exp(2\pi i z)$. For $n \in \mathbb{Z}$, we define the twisted Kloosterman sum

$$
K_c(\chi,t,n) := \sum_{d\in(\mathbb{Z}/c\mathbb{Z})^*} \chi(d)^{-1} e\left(\frac{t(d^{-1} \bmod c) + nd}{c}\right). \tag{2.1}
$$

We let $\Gamma_0(N)_\infty$ denote the stabilizer of ∞ in $\Gamma_0(N)$. For matrices $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we define the slash operator $|_k M$ (see [12] for example) on functions $f(z)$ in the usual way, and we define $\chi(M) := \chi(d)$. We also require the I- and J-Bessel functions.

Using this notation, one defines the classical holomorphic Poincaré series

$$P(t, k, N, \chi; z) := \sum_{M \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \left(\chi(M)^{-1} e(tz) \mid_k M\right). \tag{2.2}$$

For positive $m \in \mathbb{Z}$, we find that $P(m, k, N, \chi; z) \in S_k(\Gamma_0(N), \chi)$ and that its q-expansion is

$$q^m + 2\pi(-i)^k \sum_{n=1}^{\infty} \left(\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K_c(\chi, m, n)}{c} J_{k-1}\left(\frac{4\pi}{c}\sqrt{nm}\right) \right) q^n.$$

In the special case where $S_k(\Gamma_0(N), \chi)$ is one-dimensional, spanned by the normalized eigenform $g(z)$, one can show that

$$P(1, k, N, \chi; z) = \frac{(k-2)!}{(4\pi)^{k-1}} \cdot g \cdot ||g||^{-2}, \tag{2.3}$$

where $||\cdot||$ is the Petersson norm; see, for example, §3 of [11]. On the other hand, we find that $P(-m, k, N, \chi; z) \in M_k^!(\Gamma_0(N), \chi)$, the space of weakly holomorphic modular forms of weight k, level N, and character χ, and that its q-expansion is

$$q^{-m} + 2\pi(-i)^k \sum_{n=1}^{\infty} \left(\left(\frac{n}{m}\right)^{\frac{k-1}{2}} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K_c(\chi, -m, n)}{c} I_{k-1}\left(\frac{4\pi}{c}\sqrt{nm}\right) \right) q^n. \tag{2.4}$$

Next, we define certain non-holomorphic Poincaré series. Such series are harmonic weak Maass forms, which we now define. Let N, k, and χ be as above, let $x, y \in \mathbb{R}$, and let $z = x + iy \in \mathfrak{h}$, the complex upper half-plane. The weight k hyperbolic Laplacian is

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

A harmonic weak Maass form of weight $2-k$ on $\Gamma_0(N)$ is a smooth function f on $\mathfrak{h}$ satisfying:

(i) For all $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we have $(f|_k M)(z) = \chi(d) f(z)$.

(ii) $\Delta_{2-k} f = 0$.

(iii) There is a polynomial $P_f = \sum_{n \le 0} c_f^+(n) q^n \in \mathbb{C}[q^{-1}]$ such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$. One requires analogous conditions at all cusps.

For a fixed cusp, one calls the corresponding polynomial P_f in (iii) the principal part of the expansion at the cusp. We denote the vector space of forms satisfying (i)-(iii) by $H_{2-k}(\Gamma_0(N), \chi)$. Let f be a form in this space, and let $\Gamma(a, x)$ denote the incomplete Gamma-function. One can show that the q-expansion of f has the form

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(k-1, 4\pi|n|y) q^n.$$

The first summand, $f^+(z)$, is the holomorphic part of f; the second summand, $f^-(z)$, is the non-holomorphic part of f. For more details on harmonic weak Maass forms, see [13].

To define the non-holomorphic Poincaré series of interest, let $s \in \mathbb{C}$, let $y \in \mathbb{R} - \{0\}$, and let $M_{\nu,\mu}$ be the usual M-Whittaker function with parameters μ, ν. We define

$$\mathcal{M}_s(y) := |y|^{\frac{k}{2}-1} M_{\mathrm{sgn}(y)\left(1-\frac{k}{2}\right), s-\frac{1}{2}}(|y|),$$

and we define

$$\phi_s(z) := \mathcal{M}_s(4\pi y) e(x).$$

Using these objects, we define for positive integers m the non-holomorphic Poincaré series

$$\mathcal{P}(m, 2-k, N, \chi; z) := \frac{1}{(k-1)!} \sum_{M \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \left(\chi(M)^{-1} \phi_{k/2}(-mz) \mid_{2-k} M\right). \tag{2.5}$$

One finds that $\mathcal{P}(m, 2-k, N, \chi, z) \in H_{2-k}(\Gamma_0(N), \chi)$. Moreover, its q-expansion is

$$q^{-m} + \sum_{n \geq 0}^{\infty} a^+(n) q^n - \frac{1}{(k-2)!} \Gamma(k-1, 4\pi m y) q^{-m} + \sum_{n<0} a^-(n) \Gamma(k-1, 4\pi |n| y) q^n, \tag{2.6}$$

where

$$\begin{aligned} a^+(n) &= -2\pi i^k \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K_c(\chi, -m, n)}{c} I_{k-1}\left(\frac{4\pi}{c}\sqrt{nm}\right) \quad \text{for } n \geq 1; \\ a^+(0) &= -(2\pi i)^k m^{k-1} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K_c(\chi, -m, 0)}{c} c^k; \\ a^-(n) &= -\frac{2\pi i^k}{(k-2)!} m^{\frac{k-1}{2}} |n|^{\frac{1-k}{2}} \sum_{\substack{c=1 \\ N|c}}^{\infty} \frac{K_c(\chi, -m, n)}{c} J_{k-1}\left(\frac{4\pi}{c}\sqrt{|nm|}\right) \quad \text{for } n \leq -1. \end{aligned} \tag{2.7}$$

Remark 3. The Poincaré series $\mathcal{P}(m, 2-k, N, \chi; z)$ has constant principal parts at cusps inequivalent to infinity; see for example Theorem 3.2 of [3].

Certain differential operators induce useful connections between the holomorphic Poincaré series (2.2) and the non-holomorphic Poincaré series (2.5). Let ξ_{2-k} be the differential operator as in (1.1). From the definitions, one can show that

$$\xi_{2-k}(\mathcal{P}(m, 2-k, N, \chi; z)) = \frac{(4\pi)^{k-1} m^{k-1}}{(k-2)!} \cdot P(m, k, N, \overline{\chi}; z). \tag{2.8}$$

Specializing to the setting where $S_k(\Gamma_0(N), \overline{\chi})$ is one-dimensional, spanned by the normalized eigenform g, we find by (2.3) and (2.8) that

$$\xi_{2-k}(\mathcal{P}(1, 2-k, N, \chi; z)) = \frac{(4\pi)^{k-1}}{(k-2)!} \cdot P(1, k, N, \overline{\chi}; z) = g \cdot ||g||^{-2}. \tag{2.9}$$

We also consider the differential operator

$$D := q \cdot \frac{d}{dq} = \frac{1}{2\pi i} \cdot \frac{d}{dz}.$$

In [7], Bruinier, Ono, and Rhoades prove that

$$D^{k-1} : H_{2-k}(\Gamma_0(N), \chi) \to M_k^!(\Gamma_0(N), \chi). \tag{2.10}$$

Moreover, they show for all $f \in H_{2-k}(\Gamma_0(N), \chi)$, that

$$D^{k-1} f = D^{k-1} f^+.$$

In particular, for $f(z) = \mathcal{P}(m, 2-k, N, \chi; z)$ we find that

$$D^{k-1}\mathcal{P}(m, 2-k, N, \chi; z) = D^{k-1}\mathcal{P}^+(m, 2-k, N, \chi; z) = (-m)^{k-1} P(-m, k, N, \chi; z). \tag{2.11}$$

3. *Preliminary facts.*

The proof of Theorem 1.1 follows from an application of Theorem 1.2 (2) of [9]. We will state the special case of this result necessary for our proof. We first require some notation. Suppose that the character χ is quadratic and that $g(z) = \sum b_g(n) q^n$ in $S_k(\Gamma_0(N), \chi)$ is a newform with rational coefficients and complex multiplication (CM) with CM field $L = \mathbb{Q}(\sqrt{-d})$, where $d > 0$ is a square-free integer. For details on modular forms including CM forms, see [11] for example.

We say that $f(z) \in H_{2-k}(\Gamma_0(N), \chi)$ is good for $g(z)$ if it satisfies:

(i) The principal part of f at the cusp infinity is in $\mathbb{Q}[q^{-1}]$.

(ii) The principal parts of f at cusps inequivalent to infinity are constant.
(iii) We have $\xi_{2-k}(f) = g \cdot ||g||^{-2}$.

An important property of certain good forms is given by the following theorem.

THEOREM 3.1 ([7, Theorem 1.3]). *Assume the notation above. Suppose that $f(z)$ is good for the CM newform $g(z)$. Then $f^+(z)$ has q-expansion coefficients in a number field.*

Consequently, the q-expansion coefficients of $D^{k-1}f^+(z)$ must also be algebraic, lying in the same number field.

Next, we consider for all primes p the polynomial

$$X^2 - b_g(p)X + \chi(p)p^{k-1} = (X-\beta)(X-\beta') \tag{3.1}$$

with roots distinguished by $\mathrm{ord}_p(\beta) \le \mathrm{ord}_p(\beta')$. We note that

$$\beta + \beta' = -b_g(p), \tag{3.2}$$

$$\beta\beta' = \chi(p)p^{k-1}. \tag{3.3}$$

For positive integers m, we define the $U(m)$ and $V(m)$ operators on formal power series by

$$\left(\sum a(n)q^n\right) \mid U(m) := \sum a(mn)q^n, \quad \left(\sum a(n)q^n\right) \mid V(m) := \sum a(n)q^{mn}.$$

For primes p, we define the Hecke operator $T_{k,\chi}(p)$ on the space $H_k(\Gamma_0(N),\chi)$ by

$$T_{k,\chi}(p) := U(p) + \chi(p)p^{k-1}V(p). \tag{3.4}$$

We now state the theorem from [9] that we require.

THEOREM 3.2 ([9, Theorem 1.2 (2)]). *Assume the notation above. Suppose that f is good for the CM newform g, and define numbers $c(n)$ by*

$$D^{k-1}f^+(z) =: \sum_{n \gg -\infty} c(n)q^n.$$

Then for all primes p inert in the CM field L such that the p-adic limit

$$L_p := \lim_{w\to\infty} \beta^{-2w} D^{k-1}f^+ \mid U(p^{2w+1}) \tag{3.5}$$

exists and is non-zero, we must have the p-adic limit

$$g(z) = \lim_{w\to\infty} \frac{D^{k-1}f^+ \mid U(p^{2w+1})}{c(p^{2w+1})}.$$

For the proof of Theorem 1.1 we further require a preliminary lemma on the existence of weakly holomorphic modular forms with prescribed principal parts.

LEMMA 3.3. *Let $\chi_{-1}(\cdot) := \left(\frac{-1}{\cdot}\right)$, and let $j \ge 2$ be an integer. Then there is a modular form $w_j(z) \in M^!_{-1}(\Gamma_0(16),\chi_{-1}) \cap \mathbb{Z}((q))$ satisfying the following properties:*
(i) *Its q-expansion is supported on exponents $n \equiv -j \pmod 4$.*
(ii) *Its principal parts at cusps inequivalent to infinity are constant.*
(iii) *If $j \not\equiv 1 \pmod 4$, its principal part at infinity is q^{-j}; if $j \equiv 1 \pmod 4$, there exists $t_j \in \mathbb{Z}$ such that its principal part at infinity is $q^{-j} + t_j q^{-1}$.*

Proof. We first prove the existence of a form $\widetilde{w}_j$ satisfying the first and second property and the following weaker version of the third property:

$$\widetilde{w}_j - q^{-j} \in q^{-j+1}\mathbb{Z}[[q]]. \tag{3.6}$$

We use the following forms:

$$w(z) := 4 \cdot \frac{1}{\theta_2(4z)^2} = \frac{\eta(8z)^2}{\eta(16z)^4} = q^{-2} - 2q^6 + 3q^{14} + \cdots \in M^!_{-1}(\Gamma_0(16), \chi_{-1}),$$
$$h(z) := 4 \cdot \frac{\theta_3(4z)^2}{\theta_2(2z)^2} = \frac{\eta(8z)^6}{\eta(4z)^2\eta(16z)^4} = q^{-1} + 2q^3 - q^7 - 2q^{11} + \cdots \in M^!_0(\Gamma_0(16)).$$

The form $h(z)$ is a generator for the function field of the genus zero group $\Gamma_0(16)$. Using Theorem 1.65 of [12], we find that these forms have constant principal parts at cusps inequivalent to infinity. They also have integer coefficients as ratios of forms with integer coefficients whose denominator has leading coefficient one. For all integers $j \geq 2$, it follows that the forms

$$\widetilde{w}_j(z) := h(z)^{j-2}w(z) = q^{-j} + \cdots =: \sum_{n \gg -\infty} a_{\widetilde{w}_j}(n) \in M_{-1}(\Gamma_0(16), \chi_{-1}) \cap \mathbb{Z}((q))$$

satisfy (3.6) and property (ii). Moreover, since $w(z)$ has support on exponents $n \equiv 2 \pmod 4$ and $h(z)$ has support on exponents $n \equiv 3 \pmod 4$, it follows that $\widetilde{w}_j(z)$ satisfies property (i). Now define $w_2(z) := \tilde{w}_2(z)$, and for $j > 2$, iteratively define forms w_j from the forms $\widetilde{w}_j$ as follows:

$$w_j(z) := \widetilde{w}_j(z) - \sum_{\substack{k=2 \\ k \equiv j \pmod 4}}^{j-1} \alpha_{\widetilde{w}_j}(-k) \cdot w_k(z).$$

By its construction, this form satisfies all desired properties. □

4. *Proof of Theorem 1.1.*

To prove Theorem 1.1, we apply Theorem 3.2 to the CM newform

$$g(z) := \eta(4z)^6 = \sum_{n=1}^{\infty} a_g(n)q^n = q - 6q^5 + 9q^9 + 10q^{13} + \cdots \in S_3(\Gamma_0(16), \chi_{-1}) \tag{4.1}$$

with CM field $L = \mathbb{Q}(\sqrt{-1})$. In terms of the functions (1.3), (1.4), (1.8), (1.10), and (1.11), we may rewrite $g(z)$ as

$$g(z) = \frac{1}{16} \cdot \theta_3(z)\theta_4(z)\theta_2(z)^4 = \frac{1}{16} \cdot b(2z)c(z)^2 = \frac{1}{64} \cdot F\left(\frac{1}{x(2z)}\right) \cdot G(x(z/2))^2. \tag{4.2}$$

Remark 4. Our proof of Theorem 1.1 closely mirrors that of Theorem 1.3 of [9], adapted to the setting of odd weight and non-trivial character. We observe that the method applies generally to CM newforms in one-dimensional spaces. A study of other newforms with CM by $\mathbb{Q}(\sqrt{-1})$ such as $\eta(z)^4\eta(2z)^2\eta(4z)^4 \in S_5(\Gamma_0(4), \chi_{-1})$ would yield variants of Theorem 1.1.

The proof proceeds by a series of lemmas, the first of which is

Lemma 4.1. *The Poincaré series $\mathcal{P}(z) := \mathcal{P}(1, -1, 16, \chi_{-1}; z)$ is good for $g(z)$.*

Proof. By (2.6) and a short computation with (2.7), we find that the principal part of $\mathcal{P}(z)$ at infinity is q^{-1}. Moreover, Remark (3) asserts that the principal parts at cusps inequivalent to infinity are constant. Since $S_3(\Gamma_0(16), \chi_{-1})$ is one-dimensional, (2.9) implies that $\xi_{-1}(\mathcal{P}(z)) = g \cdot ||g||^{-2}$, which completes the proof of the lemma. □

Next, we relate $\mathcal{P}(z)$ to $\Omega(z)$.

LEMMA 4.2. *We have*

$$D^2\mathcal{P}^+(z) = \Omega(z) = \sum_{n=-1}^{\infty} C(n)q^n \in M_3^!(\Gamma_0(16), \chi_{-1}). \tag{4.3}$$

Proof. By (2.11), it suffices to show that $P(z) := P(-1, 3, 16, \chi_{-1}; z) = \Omega(z)$. We compare principal parts at all cusps. In view of (2.11), it follows that $P(z)$ has principal part q^{-1} at infinity and constant principal parts at other cusps since $\mathcal{P}(z)$ does. Similarly, we find by (1.13) that $\Omega(z)$ has principal part q^{-1} at infinity. Using the eta-quotient (1.14) and Theorem 1.65 of [12], we deduce that the principal parts at all other cusps vanish. Hence, the forms $P(z)$ and $\Omega(z)$ have principal parts which agree at all cusps. Therefore, the modular form $B(z)$ defined by

$$B(z) := \sum a_B(n)q^n = P(z) - \Omega(z) \in M_3^!(\Gamma_0(16), \chi_{-1}) \tag{4.4}$$

vanishes at all cusps. As such, it is a cusp form in the one-dimensional space $S_3(\Gamma_0(16), \chi_{-1})$. Since this space is spanned by the newform $g(z)$ as in (4.1), there is a constant r for which

$$B(z) = r \cdot g(z) \tag{4.5}$$

A calculation reveals for all positive integers $c \equiv 0 \pmod{16}$ and $n \equiv 1 \pmod 4$, that the Kloosterman sum (2.1) satisfies

$$K_c(\chi_{-1}, -1, n) = 0. \tag{4.6}$$

Using (1.13), (2.4), (4.4), and (4.6), we find that

$$a_B(1) = 2\pi i \sum_{\substack{c=1 \\ 16|c}}^{\infty} \frac{K_c(\chi_{-1}, -1, 1)}{c} I_2\left(\frac{4\pi}{c}\right) = 0. \tag{4.7}$$

Since $a_g(1) = 1$, we see by (4.5) and (4.7) that $r = 0$, and hence that $P(z) = \Omega(z)$. □

REMARK 5. Since $\Omega(z)$ has integer coefficients, it follows by (4.3) that $\mathcal{P}^+(z)$ has rational coefficients. In particular, we find that

$$\mathcal{P}^+(z) = \sum_{n=-1}^{\infty} c_{\mathcal{P}}^+(n)q^n = q^{-1} - \frac{2}{9}q^3 - \frac{13}{49}q^7 + \frac{26}{121}q^{11} + \frac{1}{3}q^{15} + \cdots. \tag{4.8}$$

To conclude the proof of Theorem 1.1, we finish verifying the hypotheses of Theorem 3.2.

LEMMA 4.3. *Let $C(n)$ be as in* (1.13), *and suppose that $p \equiv 3 \pmod 4$ is prime with $p^2 \nmid C(p)$. Then, with L_p as in* (3.5), *we must have $L_p \neq 0$.*

Before we prove the lemma, we note that it implies, by Theorem 3.2, the p-adic limit

$$g(z) = \lim_{w\to\infty} \frac{\Omega(z) \mid U(p^{2w+1})}{C(p^{2w+1})}.$$

Theorem 1.1 now follows from (4.2). Further, we remark that with β as in (3.1), equations (3.2) and (3.3) imply, for $g(z)$ as in (4.1), that $\beta^2 = p^2$. Therefore, we may write

$$L_p = \lim_{w\to\infty} p^{-2w}\Omega(z) \mid U(p^{2w+1}).$$

Proof Proof of Lemma 4.3. We suppose on the contrary that $L_p = 0$. To begin, we note for all primes p that $g(z)$ is an eigenform for $T_{3,\chi_{-1}}(p)$. For primes $p \equiv 3 \pmod 4$, the eigenvalue is $a_g(p) = 0$ since g has CM by L. For such primes, since $\mathcal{P}$ is good for g, a modification of Theorem 7.4 of [6] to integer weight gives

$$\mathcal{P} \mid T_{-1,\chi_{-1}}(p) - p^{-2}a_g(p)\mathcal{P} = \mathcal{P}^+ \mid T_{-1,\chi_{-1}}(p) \in M_{-1}^!(\Gamma_0(16), \chi_{-1}).$$

From (3.4) and (4.8), we find that

$$R_p(z) := \mathcal{P}^+ \mid T_{-1,\chi_{-1}}(p) = -p^{-2}q^{-p} + O(q). \tag{4.9}$$

We define for integers $n \geq 1$, numbers $a(n)$ by

$$-p^2 R_p(z) = q^{-p} + \sum_{n=1}^{\infty} a(n)q^n. \tag{4.10}$$

Further, we require

$$r_p(z) := p^2 D^2 R_p(z). \tag{4.11}$$

By (2.10), we have $r_p(z) \in M_3^!(\Gamma_0(16), \chi_{-1})$, and we observe by (4.10) and (4.11) that

$$-r_p(z) = p^2 q^{-p} + \sum_{n=1}^{\infty} n^2 a(n) q^{\cdot} \tag{4.12}$$

The commutation relation for D^{k-1} and $T_{k,\chi}(p)$ (see, for example [9]) together with (4.3) give

$$p^2 D^2(\mathcal{P}^+ \mid T_{-1,\chi_{-1}}(p)) = (D^2\mathcal{P}^+) \mid T_{3,\chi_{-1}}(p) = \Omega(z) \mid T_{3,\chi_{-1}}(p),$$

from which it follows by (4.9) and (4.11) that

$$r_p(z) = \Omega(z) \mid T_{3,\chi_{-1}}(p).$$

Recalling that $p \equiv 3 \pmod 4$, equation (3.4) implies that

$$r_p(z) = \Omega(z) \mid U(p) - p^2\Omega(pz).$$

Applying $U(p)$ twice and and dividing by p^2, we obtain

$$p^{-2}\Omega(z) \mid U(p^3) = \Omega(z) \mid U(p) + p^{-2} r_p(z) \mid U(p^2).$$

An induction on integers $w \geq 1$ shows that

$$p^{-2w}\Omega(z) \mid U(p^{2w+1}) = \Omega(z) \mid U(p) + \frac{1}{p^2} \cdot r_p(z) \mid U(p^2) + \cdots + \frac{1}{p^{2w}} \cdot r_p(z) \mid U(p^{2w}).$$

Taking the limit as $w \to \infty$ and recalling the hypothesis that $L_p = 0$, we deduce that

$$-\Omega(z) \mid U(p) = \sum_{w=1}^{\infty} p^{-2w} r_p(z) \mid U(p^{2w}).$$

Equating coefficients of q using (1.13) and (4.12) yields

$$C(p) = p^2 \sum_{w=0}^{\infty} p^{2w} a(p^{2w+2}).$$

If we can show for all $n \geq 1$ that $a(n) \in \mathbb{Z}$, then we have $p^2 \mid C(p)$, a contradiction.

Therefore, it remains to prove integrality of the coefficients $a(n)$. Let $w_p(z)$ as in Lemma 3.3. By Lemma 3.3 (iii) and (4.9), we have

$$\widetilde{\Omega}_p(z) := -w_p(z) - p^2 R_p(z) = O(q) \in M_{-1}^!(\Gamma_0(16), \chi_{-1}).$$

Moreover, Lemma 3.3 (ii) and the fact that $g(z)$ is good for $\mathcal{P}$ imply that $\widetilde{\Omega}_p(z)$ has constant principal parts at all other cusps. Hence, $\widetilde{\Omega}_p(z) \in M_{-1}(\Gamma_0(16), \chi_{-1}) = \{0\}$. In particular, we must have $w_p(z) = -p^2 R_p(z)$. Since the coefficients of $w_p(z)$ are integers, we deduce that the coefficients $a(n)$ of $-p^2 R_p(z)$ are also integers. The lemma follows, and with it, Theorem 1.1. □

References

[1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] G. E. Andrews, R. Askey, and R. Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.

[3] K. Bringmann and K. Ono, *Coefficients of harmonic weak Maass forms*, Proceedings of the 2008 University of Florida conference on partitions, q-series, and modular forms, accepted for publication.
[4] J. H. Bruinier, *Borcherds products on O(2, l) and Chern classes of Heegner divisors*, Lecture Notes in Mathematics, vol. 1780, Springer-Verlag, Berlin, 2002.
[5] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90.
[6] J. H. Bruinier and K. Ono, *Heegner divisors, L-functions, and Maass forms*, Ann. of Math., to appear.
[7] J. H. Bruinier, K. Ono, and R. C. Rhoades, *Differential operators for harmonic weak Maass forms and the vanishing of Hecke eigenvalues*, Math. Ann. **342** (2008), 673-693.
[8] J. M. Borwein and P. B. Borwein, *Pi and the AGM*, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons Inc., New York, 1987. A study in analytic number theory and computational complexity; A Wiley-Interscience Publication.
[9] P. Guerzhoy, Z. A. Kent, and K. Ono, *p-adic coupling of mock modular forms and shadows*, Proc. Natl. Acad. Sci. USA (2009), to appear.
[10] D. A. Hejhal, *The Selberg trace formula for* PSL(2, **R**). *Vol. 2*, Lecture Notes in Mathematics, vol. 1001, Springer-Verlag, Berlin, 1983.
[11] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI, 1997.
[12] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.
[13] ———, *Unearthing the visions of a master: harmonic Maass forms and number theory*, Harvard-MIT Current Developments in Mathematics, International Press, 2008, accepted for publication.
[14] R. A. Rankin, *Modular forms and functions*, Cambridge University Press, Cambridge, 1977.
[15] D. Zagier, *Ramanujan's mock theta functions and their applications [d'aprés Zwegers and Bringmann-Ono]*, Séminaire Bourbaki **986** (2006).

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