ARITHMETIC PROPERTIES OF CERTAIN LEVEL ONE MOCK MODULAR FORMS

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Abstract. In recent work, Bringmann and Ono [4] show that Ramanujan's $f(q)$ mock theta function is the holomorphic projection of a harmonic weak Maass form of weight 1/2. In this paper, we extend work of Ono in [13]. In particular, we study holomorphic projections of certain integer weight harmonic weak Maass forms on $\mathrm{SL}_2(\mathbb{Z})$ using Hecke operators and the differential theta-operator.

1. Introduction and statement of results.

If k is an integer, we denote by M_k (respectively, S_k) the space of holomorphic modular forms (respectively, cusp forms) of weight k on $\mathrm{SL}_2(\mathbb{Z})$. Let $z \in \mathfrak{h}$, the complex upper half-plane, and let $q := e^{2\pi i z}$. For integers $n \geq 1$ and $j \geq 0$, define $\sigma_j(n) := \sum_{d|n} d^j$, and let B_j denote the jth Bernoulli number. Then the Delta-function and the normalized Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ of even weight $k \geq 2$ are given by

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{m=1}^{\infty} (1-q^m)^{24} \in S_{12}, \tag{1.1}$$

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{1.2}$$

If $k \geq 4$, then $E_k(z) \in M_k$. We also set $E_0(z) := 1$. If the dimension of S_k is one, then $k \in \{12, 16, 18, 20, 22, 26\}$. We note that the corresponding weight k cusp forms,

$$f_k(z) := \sum_{n=1}^{\infty} a_{f_k}(n) q^n = \Delta(z) E_{k-12}(z), \tag{1.3}$$

have integer coefficients.

In [13], Ono defined a harmonic weak Maass form related to the Delta-function and studied its holomorphic projection. In this paper, we extend Ono's work by defining harmonic weak Maass forms related to the forms $f_k(z)$ and by studying their holomorphic projections.

We denote spaces of harmonic weak Maass forms of weight j on $\mathrm{SL}_2(\mathbb{Z})$ by H_j. In Section 2, we will use Poincaré series to define, as in [5] and [13], harmonic weak Maass forms $R_{f_k}(z) \in H_{2-k}$ connected to the forms $f_k(z) \in S_k$. Such forms may be written $R_{f_k}(z) = M_{f_k}(z) + N_{f_k}(z)$, where $M_{f_k}(z)$ is holomorphic on $\mathfrak{h}$ and $N_{f_k}(z)$ is not holomorphic on $\mathfrak{h}$. We will find that the non-holomorphic contribution is a normalization of the period integral for $f_k(z)$,

Date: July 2, 2008.
2000 *Mathematics Subject Classification.* 11F03, 11F11.
The author thanks the National Science Foundation for its support through grant DMS-0600400.

given by

$$N_{f_k}(z) = i(k-1)(2\pi)^{k-1}c_k \int_{-\overline{z}}^{i\infty} \frac{\overline{f_k(-\overline{\tau})}}{(-i(\tau+z))^{2-k}}d\tau, \tag{1.4}$$

where

$$c_k := \frac{(k-2)!}{(4\pi)^{k-1}\|f_k(z)\|^2} \in \mathbb{R}. \tag{1.5}$$

Here, $\|\cdot\|$ denotes the Petersson norm. We will also find that the function $M_{f_k}(z)$, the holomorphic projection of $R_{f_k}(z)$, has coefficients $c^+_{f_k}(n)$, and is given by

$$M_{f_k}(z) = \sum_{n=-1}^{\infty} c^+_{f_k}(n)q^n = (k-1)!q^{-1} + (k-1)!\frac{2k}{B_k} + \cdots \tag{1.6}$$

See Theorem 2.1 for precise formulas for the coefficients $c^+_{f_k}(n)$ as series with summands in terms of Kloosterman sums and values of modified Bessel functions of the first kind.

The theory of harmonic weak Maass forms and mock theta functions is of great current interest. See, for example, the works [3], [4], [5], [6], [7], [8], [9], [13]. Let $\lambda \in \{1/2, 3/2\}$. In [3] and [6], Bringmann, Ono, and Folsom define a mock theta function to be a function which arises as the holomorphic projection of a harmonic weak Maass form of weight $2-\lambda$ whose non-holomorphic part is a period integral of a linear combination of weight λ theta-series. This definition encompasses the mock theta functions of Ramanujan, of which

$$f(q) := \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}$$

is an important example. See [4] for its precise relation to Maass forms. In this sense, one may view the functions $M_{f_k}(z)$ as analogues of mock theta functions which we call mock modular forms.

An important aspect of the study of mock modular forms concerns questions on the rationality (algebraicity) of their q-expansion coefficients. Mock modular forms arising from theta functions, such as $f(q)$, are known to have rational (algebraic) q-expansion coefficients. In [8] and [9], the authors initiated a study of the algebraicity of coefficients of mock modular forms $M(z) = \sum c^+(n)q^n$ which do not arise from theta functions. In [8], Bruinier and Ono relate the algebraicity of the coefficients of certain weight 1/2 mock modular forms to the vanishing of derivatives of quadratic twists of central critical values of weight two modular L-functions. In [9], Bruinier, Ono, and Rhoades give conditions which guarantee the algebraicity of coefficients of integer weight mock modular forms. In the proof of their result, they show that the coefficients $c^+(n)$ lie in the field $\mathbb{Q}(c^+(1))$. Part 1 of Corollary 1.3 below recovers this result for the coefficients $c^+_{f_k}(n)$. However, the conditions in [9] guaranteeing the algebraicity of mock modular form coefficients do not apply in the present setting. As such, it is not known whether any $c^+_{f_k}(1)$ is algebraic.

In [13], Ono conjectures, for all positive integers n, that the mock Delta coefficients, $c^+_{\Delta}(n)$, are irrational. This conjecture has the following consequence. If $c^+_{\Delta}(1)$ (or any coefficient $c^+_{\Delta}(n)$) is irrational, then Lehmer's Conjecture on the non-vanishing of the tau function is true. Similarly, as a consequence of the work in this paper, we observe that if $c^+_{f_k}(1)$ (or any $c^+_{f_k}(n)$) is irrational, then for all positive integers n, we have $a_{f_k}(n) \neq 0$, which is the analogue of Lehmer's conjecture for the forms f_k. This is part 2 of Corollary 1.3 below.

In this paper, we use the Hecke operators and theta-operator to create certain weakly holomorphic modular forms which are modifications of the mock modular forms $M_{f_k}(z)$. We then give explicit formulas which allow us to prove results on rationality and congruence properties of the coefficients of these weakly holomorphic forms. Our formulas also enable us to prove results on the values of these functions in the upper half-plane.

To state our results, we require facts on weakly holomorphic modular forms on $\mathrm{SL}_2(\mathbb{Z})$, forms which are holomorphic on $\mathfrak{h}$, but which may have a pole at the cusp infinity. We denote the space of such forms of integer weight k by $M_k^!$ and note that $M_k \subseteq M_k^!$. For more information on holomorphic and weakly holomorphic modular forms, see [12], for example. An important example of a weakly holomorphic form is the elliptic modular invariant,

$$j(z) := \frac{E_4(z)^3}{\Delta(z)} = \sum_{n=-1}^{\infty} c(n)q^n = q^{-1} + 744 + 196884q + \cdots \in M_0^!.$$

We also require certain operators on q-expansions. If p is prime, then the pth Hecke operator in level one and integer weight k acts by

$$\sum a(n)q^n \mid T_k(p) := \sum \left(a(pn) + p^{k-1} a\left(\frac{n}{p}\right)\right) q^n, \tag{1.7}$$

where $a\left(\frac{n}{p}\right) = 0$ if $p \nmid n$. For integers $m \geq 1$, one defines Hecke operators $T_k(m)$ in terms of the Hecke operators of prime index in (1.7). Furthermore, the Hecke operators preserve the spaces M_k and $M_k^!$. Next, we define the differential theta-operator

$$\theta := \frac{1}{2\pi i} \cdot \frac{d}{dz} = q\frac{d}{dq} \tag{1.8}$$

and observe that

$$\theta\left(\sum a(n)q^n\right) = \sum na(n)q^n.$$

Applying a normalization of the Hecke operators to $j(z)$, we obtain an important sequence of modular forms. We set $j_0(z) := 1$,

$$j_1(z) := j(z) - 744 = q^{-1} + 196884q + \cdots,$$

and for all $m \geq 1$, we set

$$j_m(z) := m(j_1(z) \mid T_0(m)) = q^{-m} + \sum_{n=1}^{\infty} c_m(n)q^n \in M_0^!. \tag{1.9}$$

The forms $j_m(z)$ have integer coefficients and satisfy interesting properties. For example, let $\tau \in \mathfrak{h}$ and define $H_\tau(z) := \sum_{m=0}^{\infty} j_m(\tau)q^m$. We will need the fact, proved in [2], that $H_\tau(z)$ is a weight two meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ and that

$$H_\tau(z) = \sum_{m=0}^{\infty} j_m(\tau)q^m = -\frac{\theta(j(z) - j(\tau))}{j(z) - j(\tau)} = \frac{E_{14}(z)}{\Delta(z)} \cdot \frac{1}{j(z) - j(\tau)}. \tag{1.10}$$

To state our main result, we require further definitions. Let p be prime. The first function we study in connection with $M_{f_k}(z)$ is $L_{f_k,p}(z)$, defined by

$$\begin{aligned} L_{f_k,p}(z) &:= \frac{p^{k-1}}{(k-1)!}(M_{f_k}(z) \mid T_{2-k}(p) - p^{1-k}a_{f_k}(p)M_{f_k}(z)) \\ &= \frac{1}{(k-1)!}\sum_{n=-p}^{\infty}\left(p^{k-1}c^+_{f_k}(pn) - a_{f_k}(p)c^+_{f_k}(n) + c^+_{f_k}\left(\frac{n}{p}\right)\right)q^n. \end{aligned} \tag{1.11}$$

We then define $a_{L_{f_k,p}}(n)$ by $\sum_{n=-p}^{\infty} a_{L_{f_k,p}}(n)q^n := L_{f_k,p}(z)$. To study the functions $L_{f_k,p}(z)$, we introduce auxiliary functions. For $k \in \{12, 16, 18, 20, 22, 26\}$, let F_k be any form in M_{k-2} and define

$$F_k(z) := \sum_{n=0}^{\infty} a_{F_k}(n)q^n. \tag{1.12}$$

For all positive integers t, define

$$A_{F_k,t}(z) := \sum_{m=0}^{t} a_{F_k}(m)j_{t-m}(z) + a_{F_k}(0)\frac{2k}{B_k}\sigma_{k-1}(t) \tag{1.13}$$

with $j_n(z)$ as in (1.9).

The second function we study in connection with $M_{f_k}(z)$ is

$$\theta^{k-1}(M_{f_k}(z)) = -(k-1)!q^{-1} + \sum_{n=1}^{\infty} n^{k-1}c^+_{f_k}(n)q^n. \tag{1.14}$$

Its study also requires the introduction of auxiliary functions. Let $\overline{k}$ be the least positive residue of k modulo 12. We define $G_k(z)$ by

$$G_k(z) := \begin{cases} \frac{E_{12-\overline{k}}(z)}{\Delta(z)^{\frac{k-\overline{k}}{12}+1}} & \text{if } k \not\equiv 10 \pmod{12} \\ \\ \frac{E_{14}(z)}{\Delta(z)^{\frac{k-10}{12}+2}} & \text{if } k \equiv 10 \pmod{12}. \end{cases} \tag{1.15}$$

We note that $G_k(z) \in M^!_{-k} \cap \mathbb{Q}((q))$, and define $a_{G_k}(n) \in \mathbb{C}$ and $n_k \in \mathbb{Z}$ by

$$\sum_{n=-n_k}^{\infty} a_{G_k}(n)q^n := G_k(z) = q^{-n_k} + \cdots \tag{1.16}$$

For fixed k and integers $t \le n_k$ (with n_k as in (1.16)), we define

$$B_{G_k,t}(z) := \sum_{n=t}^{n_k} a_{G_k}(-n)j_{n-t}(z) \tag{1.17}$$

with $j_n(z)$ as in (1.9).

Our main result gives exact formulas for $L_{f_k,p}(z)$ and $\theta^{k-1}(M_{f_k}(z))$.

Theorem 1.1. *Let $k \in \{12, 16, 18, 20, 22, 26\}$, and let p be prime.*

(1) *In the notation above ((1.3), (1.11), (1.12), (1.13)), we have*

$$L_{f_k,p}(z) = \frac{A_{F_k,p}(z) - a_{f_k}(p) A_{F_k,1}(z)}{F_k(z)} \in M^!_{2-k}.$$

(2) *In the notation above ((1.6), (1.14), (1.16), (1.17)), we have*

$$\theta^{k-1}(M_{f_k}(z)) = \frac{\sum_{m=1}^{n_k} m^{k-1} c^+_{f_k}(m) B_{G_k,m}(z) - (k-1)! B_{G_k,-1}(z)}{G_k(z)} \in M^!_k.$$

Before proceeding, we illustrate Theorem 1.1 by applying it to $f_{12}(z) = \Delta(z)$. Let p be prime. With $F_{12}(z) = E_{10}(z)$ in part 1 of the theorem, we find that

$$L_{\Delta,p}(z) = \frac{j_p(z) - 264 \sum_{m=1}^{p} \sigma_9(m) j_{p-m}(z) - \frac{65520}{691}\sigma_{11}(p) - \tau(p)\left(j_1(z) - \frac{247944}{691}\right)}{E_{10}(z)}.$$

This is the main result in [13]. Using part 2 of the theorem, we obtain

$$\theta^{11}(M_\Delta(z)) = \Delta(z)(c^+_\Delta(1) - 11!(j_2(z) + 24 j_1(z) + 324)). \tag{1.18}$$

On examining the right-hand sides in Theorem 1.1, we deduce properties of the forms $L_{f_k,p}(z)$ and $\theta^{k-1}(M_{f_k}(z))$. First, we find that the coefficients of $L_{f_k,p}(z)$ are integers.

Corollary 1.2. *Let $k \in \{12, 16, 18, 20, 22, 26\}$, and let p be prime. Then for all integers n, we have*

$$a_{L_{f_k,p}}(n) = \frac{1}{(k-1)!}\left(p^{k-1} c^+_{f_k}(pn) - a_{f_k}(p) c^+_{f_k}(n) + c^+_{f_k}\left(\frac{n}{p}\right)\right) \in \mathbb{Z}.$$

A consequence of Corollary 1.2 is the following.

Corollary 1.3. *Let $k \in \{12, 16, 18, 20, 22, 26\}$.*

(1) *For all integers n, we have $c^+_{f_k}(n) \in \mathbb{Q}(c^+_{f_k}(1))$.*
(2) *If $c^+_{f_k}(1)$ (or any $c^+_{f_k}(n)$) is irrational, then for all integers n, we have $a_{f_k}(n) \neq 0$.*

As an example of part 1, for all primes p, Corollary 1.2 gives

$$c^+_{f_k}(p) = \frac{(k-1)! \cdot a_{L_{f_k,p}}(1)}{p^{k-1}} + \frac{a_{f_k}(p)}{p^{k-1}} \cdot c^+_{f_k}(1). \tag{1.19}$$

The general case follows by induction. As remarked above, the conclusion of part 2 of the corollary is the analogue of Lehmer's conjecture for the forms f_k. It is proved in Section 3.

Theorem 1.1 and Corollary 1.2 permit us to apply well-known congruence properties for coefficients of the Eisenstein series $E_k(z)$ and the cusp forms $f_k(z)$ to obtain congruences for $L_{f_k,p}(z)$.

Corollary 1.4. *Let p be prime, and let $\tau(p) = a_{f_{12}}(p)$ as in (1.1).*

(1) *If $k \in \{12, 16, 18, 20, 22, 26\}$, then we have*

$$L_{f_k,p}(z) \equiv \begin{cases} j_p(z) \pmod{24} & \text{if } p = 2 \text{ or } p \equiv 23 \pmod{24} \\ j_p(z) + 12 j_1(z) \pmod{24} & \text{if } p = 3 \text{ or } p \equiv 11 \pmod{24} \\ j_p(z) - (p+1) j_1(z) \pmod{24} & \text{if } p \geq 5. \end{cases}$$

(2) *If $k \in \{18, 22, 26\}$, then we have*

$$L_{f_k,p}(z) \equiv \begin{cases} j_p(z) - 2j_1(z) \pmod 5 & \text{if } p \equiv 1 \pmod 5 \\ j_p(z) + (p+1)(j_1(z)+2) \pmod 5 & \text{if } p \equiv 2, 3 \pmod 5 \\ j_p(z) \pmod 5 & \text{if } p \equiv 4 \pmod 5. \end{cases}$$

(3) *If $k \in \{20, 26\}$, then we have*

$$L_{f_k,p}(z) \equiv \begin{cases} j_p(z) - 2j_1(z) \pmod 7 & \text{if } p \equiv 1 \pmod 7 \\ j_p(z) + 2(p+1)(j_1(z)+1) \pmod 7 & \text{if } p \equiv 2, 4 \pmod 7 \\ j_p(z) + 3(p+1) \pmod 7 & \text{if } p \equiv 3, 5 \pmod 7 \\ j_p(z) \pmod 7 & \text{if } p \equiv 6 \pmod 7. \end{cases}$$

(4) *If $k \in \{12, 22\}$, then we have*

$$L_{f_k,p}(z) \equiv j_p(z) - \tau(p)j_1(z) + 2(p + 1 - \tau(p)) \pmod{11}.$$

(5) *We also have*

$$\begin{aligned} L_{f_{26},p}(z) &\equiv j_p(z) - p\tau(p)j_1(z) - 2(p + 1 - p\tau(p)) \pmod{13} \\ L_{f_{18},p}(z) &\equiv j_p(z) - a_{f_{18}}(p)j_1(z) + 7(p + 1 - a_{f_{18}}(p)) \pmod{17} \\ L_{f_{20},p}(z) &\equiv j_p(z) - a_{f_{20}}(p)j_1(z) + 5(p + 1 - a_{f_{20}}(p)) \pmod{19}. \end{aligned}$$

Remark. In [13], Ono proved part 1 of Theorem 1.1 and Corollaries 1.2, 1.3, and 1.4 for $f_{12}(z) = \Delta(z) \in S_{12}$. We also remark that Guerzhoy has proved related congruences. See Theorem 2 and Corollary 1 of [10].

Next, we study values of $L_{f_k,p}(z)$ and $\theta^{k-1}(M_{f_k}(z))$ at points in the upper half-plane. If $g(z)$ is a function on $\mathfrak{h}$ and $\tau \in \mathfrak{h}$, then we define $v_\tau(g(z))$ to be the order of vanishing of $g(z)$ at τ. Recalling (1.2), (1.10), and (1.12), we define a meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight k with coefficients $b_{F_k,\tau}(n) \in \mathbb{C}$ by

$$\sum_{n=0}^{\infty} b_{F_k,\tau}(n)q^n := H_\tau(z)F_k(z) - a_{F_k}(0)E_k(z).$$

Similarly, recalling (1.10) and (1.15), we define a meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ of weight $2 - k$ with coefficients $\beta_{G_k,\tau}(n) \in \mathbb{C}$ by

$$\sum_{n=-n_k}^{\infty} \beta_{G_k,\tau}(n)q^n := H_\tau(z)G_k(z).$$

Corollary 1.5. *Let $k \in \{12, 16, 18, 20, 22, 26\}$, let $\tau \in \mathfrak{h}$, and let p be prime.*

(1) *We have*

$$L_{f_k,p}(\tau) = \frac{b_{F_k,\tau}(p) - a_{f_k}(p)b_{F_k,\tau}(1)}{F_k(\tau)}$$

and

$$\theta^{k-1}(M_{f_k}(\tau)) = \frac{\sum_{m=1}^{n_k} m^{k-1}c^+_{f_k}(m)\beta_{G_k,\tau}(-m) - (k-1)!\beta_{G_k,\tau}(1)}{G_k(\tau)}.$$

(2) *If* $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$, *then we have*

$$v_\omega(L_{f_k,p}(z)) \geq \begin{cases} 2 & \text{if } k \equiv 0 \pmod 6 \\ 1 & \text{if } k \equiv 4 \pmod 6 \end{cases}$$

and

$$v_\omega(\theta^{k-1}(M_{f_k}(z))) \geq \begin{cases} 2 & \text{if } k \equiv 2 \pmod 6 \\ 1 & \text{if } k \equiv 4 \pmod 6). \end{cases}$$

(3) *We have*

$$v_i(L_{f_k,p}(z)) \geq 1 \quad \text{if } k \equiv 0 \pmod 4$$

and

$$v_i(\theta^{k-1}(M_{f_k}(z))) \geq 1 \quad \text{if } k \equiv 2 \pmod 4).$$

Remark. Setting $F_k(z) = E_{k-2}(z)$ in Theorem 1.1, part 2 of the corollary implies, for $k \equiv 0, 4 \pmod 6$, that

$$a_{f_k}(p) = \frac{j_p(\omega) - \frac{2(k-2)}{B_{k-2}} \sum_{m=1}^{p} \sigma_{k-3}(m) j_{p-m}(\omega) + \frac{2k}{B_k}\sigma_{k-1}(p)}{-744 - \frac{2(k-2)}{B_{k-2}} + \frac{2k}{B_k}}.$$

Similarly, part 3 of the corollary implies, for $k \equiv 0 \pmod 4$, that

$$a_{f_k}(p) = \frac{j_p(i) - \frac{2(k-2)}{B_{k-2}} \sum_{m=1}^{p} \sigma_{k-3}(m) j_{p-m}(i) + \frac{2k}{B_k}\sigma_{k-1}(p)}{984 - \frac{2(k-2)}{B_{k-2}} + \frac{2k}{B_k}}.$$

Guerzhoy proved these identities independently in [10] (Theorem 3).

Corollary 1.3 underscores the significance of the coefficients $c^+_{f_k}(1)$. Using Corollary 1.5, we provide alternative expressions for these coefficients. Perhaps they might shed some light on questions of rationality and algebraicity.

Corollary 1.6. *The following are true.*

(1) *Set* $\tau := \begin{cases} \omega & \text{if } k \equiv 2, 4 \pmod 6 \\ i & \text{if } k \equiv 2 \pmod 4). \end{cases}$ *Let* n_k *be as in* (1.16). *Then we have*

$$c^+_{f_k}(1) = \frac{(k-1)! B_{G_k,-1}(\tau) - \sum_{m=2}^{n_k} m^{k-1} c^+_{f_k}(m) B_{G_k,m}(\tau)}{B_{G_k,1}(\tau)}.$$

(2) *There is a* $\tau \in \mathfrak{h}$ *with* $v_\tau(\theta^{11}(M_\Delta(z)) \geq 1$. *For such* τ, *we have*

$$c^+_\Delta(1) = 11!(j_2(\tau) + 24 j_1(\tau) + 324).$$

Part 1 of the corollary follows from Theorem 1.1 and parts 2 and 3 of Corollary 1.5. Part 2 is proved in Section 3.

The plan for Sections 2 and 3 of the paper is as follows. In Section 2.1, we provide the necessary background on harmonic weak Maass forms. In Section 2.2, we construct certain Poincaré series and use them to define the functions $M_{f_k}(z)$. In Section 3, we prove Theorem 1.1 and its corollaries.

2. Harmonic weak Maass forms and Poincaré Series.

2.1. **Harmonic weak Maass forms.** We recall the definition of integer weight harmonic weak Maass forms on $\mathrm{SL}_2(\mathbb{Z})$. Let $k \in \mathbb{Z}$, let $x, y \in \mathbb{R}$, and let $z = x + iy \in \mathfrak{h}$. The weight k hyperbolic Laplacian is defined by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

A harmonic weak Maass form of weight k on $\mathrm{SL}_2(\mathbb{Z})$ is a smooth function f on $\mathfrak{h}$ satisfying the following properties.

(1) For all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and all $z \in \mathfrak{h}$, we have

$$f(Az) = (cz + d)^k f(z).$$

(2) We have $\Delta_k f = 0$.

(3) There is a polynomial $P_f = \sum_{n \leq 0} c_f^+(n) q^n \in \mathbb{C}[[q^{-1}]]$ such that $f(z) - P_f(z) = O(e^{-\varepsilon y})$ as $y \to \infty$ for some $\varepsilon > 0$.

Using these conditions, one can show that harmonic weak Maass forms have q-expansions of the form

$$f(z) = \sum_{n=n_0}^{\infty} c_f^+(n) q^n + \sum_{n=1}^{\infty} c_f^-(n) \Gamma(k-1, 4\pi n y) q^{-n}, \tag{2.1}$$

where $n_0 \in \mathbb{Z}$ and the incomplete Gamma function is given by

$$\Gamma(a, x) := \int_x^{\infty} e^{-t} t^a \frac{dt}{t}.$$

For more information on this and other special functions used in this section, see, for example, [1]. We refer to the first (respectively, second) sum in (2.1) as the holomorphic (respectively, non-holomorphic) projection of $f(z)$. We also observe that $M_k^! \subseteq H_k$ and that the Hecke operators preserve H_k.

2.2. **Poincaré Series.** We now construct Poincaré series as in [5]. Let k be an integer, let $z \in \mathfrak{h}$, and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. For functions $f : \mathfrak{h} \to \mathbb{C}$, we define

$$(f \mid_k A)(z) := (cz + d)^{-k} f(Az).$$

Now, let $m \in \mathbb{Z}$, and let $\varphi_m : \mathbb{R}^+ \to \mathbb{C}$ be a function which satisfies $\varphi_m(y) = O(y^\alpha)$ as $y \to 0$ for some $\alpha \in \mathbb{R}$. If $\beta \in \mathbb{R}$, we set $e(\beta) := e^{2\pi i \beta}$. In this notation, we define

$$\varphi_m^*(z) := \varphi_m(y) e(mx).$$

Noting that $\varphi_m^*(z)$ is invariant under the action of $\Gamma_\infty := \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \ : \ n \in \mathbb{Z} \right\}$, we define the Poincaré series

$$P(m, k, \varphi_m; z) := \sum_{A \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (\varphi_m^* \mid_k A)(z).$$

We will study specializations of this series.

We first consider the Poincaré series $P(1,k,1;z)$. It is well-known that $P(1,k,1;z) \in S_k$. Since S_k has dimension one for $k \in \{12,16,18,20,22,26\}$, the form $P(1,k,1;z)$ is a constant multiple of $f_k(z)$. Moreover, we observe that

$$P(1,k,1;z) = c_k f_k(z), \tag{2.2}$$

with c_k as in (1.5). These facts may be found, for example, in §3.3 of [11].

Next, we consider certain non-holomorphic Poincaré series. Let $M_{\nu,\mu}(z)$ be the M-Whittaker function. For $k \in \mathbb{Z}$ and $s \in \mathbb{C}$, define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn}(y), s-\frac{1}{2}}(|y|),$$

and for integers $m \geq 1$, set $\varphi_{-m}(z) := \mathcal{M}_{1-\frac{k}{2}}(-4\pi m y)$. Then, for integers $k > 2$, we define the Poincaré series $R_{f_k}(z) := P(-1, 2-k, \varphi_{-1}; z)$. Recalling the definition (1.4) of $N_{f_k}(z)$ and using (2.2), we have

$$N_{f_k}(z) = i(k-1)(2\pi)^{k-1} c_k \int_{-\overline{z}}^{i\infty} \frac{\overline{f_k(-\overline{\tau})}}{(-i(\tau+z))^{2-k}} d\tau = i(k-1)(2\pi)^{k-1} \int_{\overline{z}}^{i\infty} \frac{\overline{P(1,k,1,-\overline{\tau})}}{(-i(\tau+z))^{2-k}} d\tau.$$

We now define the series $M_{f_k}(z)$ by

$$M_{f_k}(z) := R_{f_k}(z) - N_{f_k}(z).$$

We require further special functions. For $j \in \mathbb{Z}$, we denote by I_j the I_j-Bessel function. For m, n, $c \in \mathbb{Z}$ with $c > 0$, the Kloosterman sum $K(m,n,c)$ is given by

$$K(m,n,c) := \sum_{v \in (\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{m\overline{v} + nv}{c}\right), \tag{2.3}$$

where $\overline{v}$ is $v^{-1} \pmod{c}$. We now state the main theorem of this section. It is a special case of Theorem 1.1 of [5]. It asserts that $M_{f_k}(z)$ is a mock modular form for $f_k(z)$ and provides formulas for its coefficients and the coefficients of $N_{f_k}(z)$.

Theorem 2.1. *Let $k \in \{12,16,18,20,22,26\}$.*

(1) *In the notation above, we have $R_{f_k}(z) = M_{f_k}(z) + N_{f_k}(z) \in H_{2-k}$.*

(2) *The function $M_{f_k}(z)$ is holomorphic with q-series expansion*

$$M_{f_k}(z) = (k-1)!q^{-1} + (k-1)!\frac{2k}{B_k} + \sum_{n=1}^{\infty} c_{f_k}^{+}(n) q^n,$$

where for integers $n \geq 1$, we have

$$c_{f_k}^{+}(n) = -2\pi i^k n^{-\frac{k-1}{2}} \sum_{c=1}^{\infty} \frac{K(-1,n,c)}{c} I_{k-1}\left(\frac{4\pi\sqrt{n}}{c}\right).$$

(3) *The function $N_{f_k}(z)$ is non-holomorphic with q-series expansion*

$$N_{f_k}(z) = \sum_{m=1}^{\infty} c_{f_k}^{-}(m) \Gamma(k-1, 4\pi m y) q^{-m},$$

where for integers $m \geq 1$, we have

$$c_{f_k}^{-}(m) = -(k-1)c_k \frac{a_{f_k}(m)}{m^{k-1}},$$

with $a_{f_k}(m) \in \mathbb{Z}$ as in (1.3) and $c_k \in \mathbb{R}$ as in (1.5).

Remark 1. The functions $M_{f_k}(z)$ depend on the representation of $f_k(z)$ as a linear combination of Poincaré series $P(m,k,1;z) \in S_k$. In this work, we choose the simplest representation, given by (2.2).

Remark 2. We compute the constant term in the q-expansion of $M_{f_k}(z)$ using (2.3) and facts from elementary number theory. In [5], we find that the constant term is

$$c^+_{f_k}(0) = -(2\pi i)^k \sum_{c=1}^{\infty} \frac{K_{2-k}(-1,0,c)}{c^k}.$$

We compute that

$$K_{2-k}(-1,0,c) = \sum_{v\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(-\frac{m\overline{v}}{c}\right) = \sum_{v\in(\mathbb{Z}/c\mathbb{Z})^*} e\left(\frac{mv}{c}\right) = \mu(c),$$

where μ is the Möbius function. Thus, since k is even, we have

$$\sum_{c=1}^{\infty} \frac{K_{2-k}(-1,0,c)}{c^k} = \sum_{c=1}^{\infty} \frac{\mu(c)}{c^k} = \frac{1}{\zeta(k)} = -2\cdot\frac{k!}{(2\pi i)^k B_k},$$

where ζ is the Riemann zeta-function.

Remark 3. We obtain the expansion for $N_{f_k}(z)$ from (1.4) as in the proof of Theorem 1.1 of [5].

3. Proofs of Theorem 1.1 and its corollaries.

We now turn to the proofs of Theorem 1.1 and its corollaries. The basic idea is as follows. We use the Hecke operators and theta operator to eliminate $N_{f_k}(z)$, the non-holomorphic part of the harmonic weak Maass form $R_{f_k}(z)$. This is where we use the fact that the forms f_k lie in 1-dimensional spaces. What remains is a modification of the mock modular form $M_{f_k}(z)$ which is a weakly holomorphic modular form with an explicit principal part. Since $\mathrm{SL}_2(\mathbb{Z})$ has genus one and only one cusp, one can express the weakly holomorphic modular form explicitly in terms of certain polynomials in j (Faber polynomials).

To begin, we note that for all integers m and for $k \in \{12,16,18,20,22,26\}$, the forms $f_k(z)$ are normalized eigenforms for the Hecke operators $T_k(m)$ since the dimension of S_k is one. Therefore, for all primes p and for all integers $n \geq 1$, from (1.7) we have

$$a_{f_k}(pn) + p^{k-1} a_{f_k}\left(\frac{n}{p}\right) = a_{f_k}(p)a_{f_k}(n). \tag{3.1}$$

Next, we compute $N_{f_k}(z) \mid T_{2-k}(p)$. To do so, we recall the definitions of the $U(p)$ and $V(p)$ operators on functions $f : \mathfrak{h} \to \mathbb{C}$:

$$f(z) \mid U(p) := \frac{1}{p}\sum_{a=0}^{p-1} f\left(\frac{z+a}{p}\right), \tag{3.2}$$

$$f(z) \mid V(p) := f(pz). \tag{3.3}$$

We also recall, for all integers j, that the Hecke operators $T_j(p)$ may be written as

$$f(z) \mid T_j(p) = f(z) \mid U(p) + p^{j-1} f(z) \mid V(p). \tag{3.4}$$

We now state two important facts. Though the first was proved in [8] for half-integral weights, (see Theorem 7.4), we provide a short proof in our setting.

Proposition 3.1. *Let p be prime, and let $k \in \{12, 16, 18, 20, 22, 26\}$. Then we have*

$$N_{f_k}(z) \mid T_{2-k}(p) = p^{1-k} a_{f_k}(p) N_{f_k}(z),$$

with $a_{f_k}(n)$ as in (1.3).

Proof. Using part 3 of Theorem 2.1, (3.1), (3.2), (3.3), and (3.4), we compute:

$$\begin{aligned}
N_{f_k}(z) \mid T_{2-k}(p) &= N_{f_k}(z) \mid U_{2-k}(p) + p^{1-k} N_{f_k}(z) \mid V_{2-k}(p) \\
&= -(k-1)c_k \left(\sum_{n=1}^{\infty} \frac{a_{f_k}(pn)}{(pn)^{k-1}} \Gamma(k-1, 4\pi ny) q^{-n} + p^{1-k} \sum_{n=1}^{\infty} \frac{a_{f_k}(n)}{n^{k-1}} \Gamma(k-1, 4\pi pny) q^{-pn} \right) \\
&= -(k-1)c_k p^{1-k} \sum_{n=1}^{\infty} \left(a_{f_k}(pn) + p^{k-1} a_{f_k}\left(\frac{n}{p}\right) \right) \frac{\Gamma(k-1, 4\pi ny)}{n^{k-1}} q^{-n} \\
&= -(k-1)c_k p^{1-k} a_{f_k}(p) \sum_{n=1}^{\infty} \frac{a_{f_k}(n)}{n^{k-1}} \Gamma(k-1, 4\pi ny) q^{-n}.
\end{aligned}$$

□

The second fact we require is a special case of a result of Bruinier, Ono, and Rhoades [9] which follows from Bol's identity relating the theta-operator and the Maass raising operator.

Theorem 3.2. *If $k \geq 2$ is an integer and $f(z) \in H_{2-k}$ has holomorphic projection $M(z) = \sum c^+(n) q^n$, then we have*

$$\theta^{k-1}(f(z)) = \theta^{k-1}(M(z)) = \sum n^{k-1} c^+(n) q^n \in M_k^!.$$

Remark. We emphasize that in [9], the authors precisely compute the image of θ^{k-1} on spaces of harmonic weak Maass forms of weight $2-k < 0$ on subgroups of type $\Gamma_0(N)$ with nebentypus character.

3.1. **Proof of Theorem 1.1.** We may now prove Theorem 1.1. From part 1 of Theorem 2.1 and the fact that the Hecke operators preserve H_{2-k}, we see that

$$L_{f_k,p}(z) := \frac{p^{k-1}}{(k-1)!} (R_{f_k}(z) \mid T_{2-k}(p) - p^{1-k} a_{f_k}(p) R_{f_k}(z)) \in H_{2-k}.$$

Then, by Proposition 3.1, part 2 of Theorem 2.1, and (1.7), we find that

$$\begin{aligned}
L_{f_k,p}(z) &= \frac{p^{k-1}}{(k-1)!} (M_{f_k}(z) \mid T_{2-k}(p) - p^{1-k} a_{f_k}(p) M_{f_k}(z)) \\
&= q^{-p} - a_{f_k}(p) q^{-1} + \frac{2k}{B_k} (\sigma_{k-1}(p) - a_{f_k}(p)) \\
&\quad + \frac{1}{(k-1)!} \sum_{n=1}^{\infty} \left(p^{k-1} c_{f_k}^+(pn) - a_{f_k}(p) c_{f_k}^+(n) + c_{f_k}^+\left(\frac{n}{p}\right) \right) q^n. \qquad (3.5)
\end{aligned}$$

Moreover, we observe that $L_{f_k,p}(z) \in M_{2-k}^!$ is a weakly holomorphic modular form since it is a harmonic weak Maass form whose holomorphic projection is zero.

With $F_k(z) = \sum_{n=0}^{\infty} a_{F_k}(n)q^n \in M_{k-2}$, we note that $L_{f_k,p}(z)F_k(z) \in M_0^!$. Using (3.5), we compute

$$\begin{aligned} L_{f_k,p}(z)F_k(z) = &\sum_{m=0}^{p} a_{F_k}(m)q^{m-p} + a_{F_k}(0)\frac{2k}{B_k}\sigma_{k-1}(p) \\ &- a_{f_k}(p)\left(a_{F_k}(0)q^{-1} + a_{F_k}(1) + a_{F_k}(0)\frac{2k}{B_k}\right) + O(q). \end{aligned}$$

If we define $A_{F_k,t}(z)$ as in (1.13), we find that

$$L_{f_k,p}(z)F_k(z) - (A_{F_k,p}(z) - a_{f_k}(p)A_{F_k,1}(z)) = O(q)$$

is a modular form of weight zero which is holomorphic on $\mathfrak{h}$ and at infinity. Hence, it is constant. This constant is zero, thus completing the proof of part 1 of Theorem 1.1.

Part 2 follows similarly. With $G_k(z) \in M_{-k}^!$ as in (1.15), we see from Theorem 3.2 that $\theta^{k-1}(M_{f_k}(z))G_k(z) \in M_0^!$. Using (1.14) and (1.16), we compute

$$\theta^{k-1}(M_{f_k}(z))G_k(z) = -(k-1)!\sum_{n=-1}^{n_k} a_{G_k}(-n)q^{-n-1} + \sum_{m=1}^{n_k} m^{k-1}c_{f_k}^{+}(m)\sum_{n=m}^{n_k} a_{G_k}(-n)q^{m-n} + O(q).$$

If we define $B_{G_k,t}(z)$ as in (1.17), we find that

$$\theta^{k-1}(M_{f_k}(z))G_k(z) - \left(\sum_{m=1}^{n_k} m^{k-1}c_{f_k}^{+}(m)B_{G_k,m}(z) - (k-1)!B_{G_k,-1}(z)\right) = O(q)$$

is a modular form of weight zero which is holomorphic on $\mathfrak{h}$ and at infinity. The conclusion follows.

3.2. **Proof of Corollaries 1.2 and 1.3.** To prove Corollary 1.2, it suffices to show that $L_{f_k,p}(z) \in \mathbb{Z}((q))$. We consider cases.

When $k \in \{12, 16\}$, we apply Theorem 1.1 with $F_k(z) = E_{k-2}(z) \in M_{k-2} \cap \mathbb{Z}[[q]]$. Here, we note that $\frac{2(k-2)}{B_{k-2}} \in \mathbb{Z}$. Furthermore, we observe that $B_{12} = -\frac{691}{2730}$, $B_{16} = -\frac{3617}{510}$, and that for all primes p,

$$\begin{aligned} \sigma_{11}(p) &\equiv a_{f_{12}}(p) = \tau(p) \pmod{691} \\ \sigma_{15}(p) &\equiv a_{f_{16}}(p) \pmod{3617}. \end{aligned}$$

These congruences may be found, for example, in [14]. Therefore, for $k \in \{12, 16\}$, we must have

$$\frac{2k}{B_k}(\sigma_{k-1}(p) - a_{f_k}(p)) \in \mathbb{Z}. \tag{3.6}$$

Hence, from (1.2), (1.13), (3.6), and Theorem 1.1, we deduce that

$$L_{f_k,p}(z) = \frac{A_{E_{k-2},p}(z) - a_{f_k}(p)A_{E_{k-2},1}(z)}{E_{k-2}(z)} \tag{3.7}$$

$$= \frac{j_p(z) - \frac{2(k-2)}{B_{k-2}}\sum_{m=1}^{p}\sigma_{k-3}(m)j_{p-m}(z) - a_{f_k}(p)\left(j_1(z) - \frac{2(k-2)}{B_{k-2}}\right) + \frac{2k}{B_k}(\sigma_{k-1}(p) - a_{f_k}(p))}{E_{k-2}(z)}$$

has integer coefficients.

When $k \in \{18, 20, 22\}$, we apply Theorem 1.1 with $F_k(z) = \Delta(z)E_{k-14}(z) \in S_k \cap \mathbb{Z}[[q]]$, from which we obtain

$$L_{f_k,p}(z) = \frac{\sum_{m=1}^{p} a_{F_k}(m) j_{p-m}(z) - a_{f_k}(p)}{\Delta(z)E_{k-14}(z)} \in \mathbb{Z}((q)).$$

Lastly, when $k = 26$, we apply Theorem 1.1 with $F_{26}(z) = \Delta(z)^2 \in S_{24} \cap \mathbb{Z}[[q]$, which gives

$$L_{f_{26},p}(z) = \frac{\sum_{m=2}^{p} a_{F_k}(m) j_{p-m}(z)}{\Delta(z)^2} \in \mathbb{Z}((q)).$$

To see part 2 of Corollary 1.3, we first remark that if $a_{f_k}(p) \neq 0$ for all primes p, then $a_{f_k}(n) \neq 0$ for all n. Therefore, we suppose that $c^+_{f_k}(m)$ is irrational for some m and that $a_{f_k}(p) = 0$ for some prime p. Corollary 1.2 implies that for all n,

$$\frac{1}{(k-1)!} \cdot \left(p^{k-1} c^+_{f_k}(pn) + c^+_{f_k}\left(\frac{n}{p}\right)\right) \in \mathbb{Z}.$$

Setting $n = 1$, we see that $c^+_{f_k}(p) \in \mathbb{Q}$. But then (1.19) shows that $c^+_{f_k}(1) \in \mathbb{Q}$. Part 1 of the corollary now implies that $c^+_{f_k}(m) \in \mathbb{Q}(c^+_{f_k}(1))$ must be rational, a contradiction.

3.3. **Proof of Corollary 1.4.** On applying Theorem 1.1 with $F_k(z) = E_{k-2}(z) \in M_{k-2}$, we obtain $L_{f_k,p}(z)$ as in (3.7). We then use the von Staudt-Claussen congruences for the denominators of Bernoulli numbers (see [12], Lemma 1.22, for example) and congruences for the coefficients $a_{f_k}(p)$ (see [14], Corollary to Theorem 4, for example) to prove Corollary 1.4.

To begin, we note that for all $j \geq 2$ and even, we have $\frac{2j}{B_j} \equiv 0 \pmod{24}$, so $E_j(z) \equiv 1 \pmod{24}$. Moreover, for $k \in \{12, 16, 18, 20, 22, 26\}$, we have $f_k(z) \equiv \Delta(z) \pmod{24}$. Part 1 of the corollary holds since for all primes p, we have

$$\tau(p) = a_{f_{12}}(p) \equiv \begin{cases} 0 \pmod{24} & \text{if } p = 2 \\ 12 \pmod{24} & \text{if } p = 3 \\ p + 1 \pmod{24} & \text{if } p \geq 5. \end{cases}$$

Next, we note that if ℓ is prime and $\ell - 1 \mid k - 2$, then we have $\frac{2(k-2)}{B_{k-2}} \equiv 0 \pmod{\ell}$, so $E_{k-2}(z) \equiv 1 \pmod{\ell}$. From (3.7), we see that if $\ell - 1 \mid k - 2$, then

$$L_{f_k,p}(z) \equiv j_p(z) - a_{f_k}(p) j_1(z) + \frac{2k}{B_k}(p + 1 - a_{f_k}(p)) \pmod{\ell}. \tag{3.8}$$

When $\ell \geq 5$, this is the case for pairs

$$(\ell, k) \in \{(5, 18), (5, 22), (5, 26), (7, 20), (7, 26), (11, 12), (11, 22), (13, 26), (17, 18), (19, 20)\}.$$

The remaining parts of Corollary 1.4 are specializations of (3.8) using congruences for $f_k(z)$, with θ as in (1.8), and congruences for $\tau(p)$ when p is prime. These congruences are:

$$f_k(z) \equiv \begin{cases} \theta\Delta(z) \pmod{5} & \text{if } k \in \{18, 22, 26\} \\ \theta\Delta(z) \pmod{7} & \text{if } k \in \{20, 26\} \\ \Delta(z) \pmod{11} & \text{if } k \in \{12, 22\} \\ \theta\Delta(z) \pmod{13} & \text{if } k = 26, \end{cases} \tag{3.9}$$

$$\tau(p) \equiv \begin{cases} p + p^2 \pmod{5} \\ p + p^4 \pmod{7}. \end{cases} \tag{3.10}$$

To prove the congruences for $f_k(z)$ modulo 5, 7, and 13, we note that $\theta\Delta(z) = \Delta(z)E_2(z)$ and that if $k \equiv k' \pmod{\ell - 1}$, then Kummer's congruences for Bernoulli numbers imply that $E_k(z) \equiv E_{k'}(z) \pmod{\ell}$.

For $k \in \{18, 22, 26\}$, (3.9) and (3.10) imply that

$$a_{f_k}(p) \equiv p\tau(p) \equiv p(p + p^2) \equiv \left(\frac{p}{5}\right)(p+1) \pmod 5,$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. Using that $\frac{2k}{B_k} \equiv 1 \pmod 5$, we obtain part 2 of the corollary. For $k \in \{20, 26\}$, (3.9) and (3.10) imply that

$$a_{f_k}(p) \equiv p\tau(p) \equiv p(p + p^4) \equiv p^2\left(1 + \left(\frac{p}{7}\right)\right) \pmod 7.$$

Using the fact that $\frac{2k}{B_k} \equiv 3 \pmod 7$ gives part 3 of the corollary. For $k \in \{12, 22\}$, (3.9) together with the fact that $\frac{2k}{B_k} \equiv 2 \pmod{11}$ gives part 4 of the corollary. Part 5 of the corollary is similar.

3.4. **Proof of Corollaries 1.5 and 1.6.** Recalling (1.2), (1.10), (1.12), and (1.13), for $n \geq 1$, we find that the nth coefficient of

$$\sum_{n=0}^{\infty} b_{F_k,\tau}(n)q^n = H_\tau(z)F_k(z) - a_{F_k}(0)E_k(z)$$

is given by

$$b_{F_k,\tau}(n) = \sum_{m=0}^{n} a_{F_k}(m)j_{n-m}(\tau) + a_{F_k}(0)\frac{2k}{B_k}\sigma_{k-1}(n) = A_{F_k,n}(z).$$

Similarly, from (1.10), (1.16), and (1.17), we find that the nth coefficient of

$$\sum_{n=-n_k}^{\infty} \beta_{G_k,\tau}(n)q^n = H_\tau(z)G_k(z)$$

is given by

$$\beta_{G_k,\tau}(n) = \sum_{m=-n_k}^{n} a_{G_k}(m)j_{n-m}(\tau) = \sum_{m=-n}^{n_k} a_{G_k}(-m)j_{m+n}(\tau) = B_{G_k,-n}(\tau).$$

Theorem 1.1 now implies part 1 of Corollary 1.5.

Next, noting from (3.5) that

$$L_{f_k,p}(z) = q^{-p} + \cdots \in M^!_{2-k},$$

we find that

$$L_{f_k,p}(z)\Delta(z)^p = \frac{\Delta(z)^p(A_{F_k,p}(z) - a_{f_k}(p)A_{F_k,1}(z))}{F_k(z)} = 1 + \cdots \in M_{12p+2-k}.$$

Applying the valence formula for forms in $M_{12p+2-k}$ (see [12], Theorem 1.29), we see that

$$v_\omega(L_{f_k,p}(z)) \equiv 2(k+1) \pmod 3,$$
$$v_i(L_{f_k,p}(z)) \equiv \frac{k}{2} + 1 \pmod 2.$$

Since these quantities are non-negative ($L_{f_{k,p}}(z)$ is weakly holomorphic), we obtain parts 2 and 3 of the corollary. Similarly, observing from (1.14) that

$$\theta^{k-1}(M_{f_k}(z))\Delta(z) = -(k-1)! + \cdots \in M_{k+12},$$

we apply the valence formula for forms in M_{k+12} to deduce parts 2 and 3 of the corollary for the weakly holomorphic modular form $\theta^{k-1}(M_{f_k}(z))$.

To obtain part 2 of Corollary 1.6, we first note that for all $\tau \in \mathfrak{h}$ we have $v_\tau(\Delta(z)) = 0$. Then applying the valence formula to

$$\theta^{11}(M_\Delta(z)) \cdot \Delta(z) = -11! + \cdots \in M_{24}$$

gives

$$\frac{1}{2} \cdot v_i(\theta^{11}(M_\Delta(z)) + \frac{1}{3} \cdot v_\omega(\theta^{11}(M_\Delta(z)) + \sum_P v_P(\theta^{11}(M_\Delta(z)) = 2,$$

where the sum is over all $P \in \mathfrak{h}$ inequivalent to i, ω in a fundamental domain. Since $\theta^{11}(M_\Delta(z))$ is weakly holomorphic, for all $\tau \in \mathfrak{h}$, we have $v_\tau(\theta^{11}(M_\Delta(z)) \geq 0$. Hence, for some $\tau \in \mathfrak{h}$, we must have $v_\tau(\theta^{11}(M_\Delta(z)) \geq 1$. Since $\Delta(\tau) \neq 0$, (1.18) implies the result.

Acknowledgment

The author thanks K. Ono for helpful comments and suggestions in the preparation of this paper.

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