# EXCEPTIONAL CONGRUENCES FOR THE COEFFICIENTS OF CERTAIN ETA-PRODUCT NEWFORMS

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## 1. Introduction and Statement of Results

In this note, we prove congruences for certain kinds of partition functions. If $r$ is a non-zero integer, we define the functions $p_r(n)$ by

$$\sum_{n=0}^{\infty} p_r(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^r. \tag{1}$$

Letting $p_{e,r}(n)$ and $p_{o,r}(n)$ denote the number of $r$-colored partitions into an even (respectively, odd) number of distinct parts, it is easy to see that

$$p_r(n) = p_{e,r}(n) - p_{o,r}(n), \tag{2}$$

when $r$ is a positive integer. Moreover, it is well-known that $p_{-1}(n) = p(n)$, the unrestricted partition function.

The functions $p_r(n)$ have been studied extensively by Ramanujan [B-O], Newman [N1], Atkin [A], and Serre [Se1]. These functions enjoy certain congruence properties. For example, the coefficients of Ramanujan's Delta-function, $\Delta(z) = \sum_{n=1}^{\infty} \tau(n)q^n$, are given by $\tau(n) = p_{24}(n-1)$ when $n \geq 1$, and they satisfy the congruence

$$p_{24}(n-1) \equiv \sigma_{11}(n) \pmod{691}, \tag{3}$$

where $\sigma_k(n) = \sum_{\substack{d|n \\ d>0}} d^k$.

Here we prove congruences of a much different flavor for $p_{12}(n)$ and the function $v(n)$ which we now define:

$$\sum_{n=0}^{\infty} v(n)q^n := \prod_{n=1}^{\infty}(1-q^n)^4(1-q^{2n})^4. \tag{4}$$

It follows that

$$v(n) = v_e(n) - v_o(n), \tag{5}$$

where $v_e(n)$ and $v_o(n)$ are the number of 8-colored partitions into an even (respectively, odd) number of distinct parts, where each of the parts of the last four colors are even.

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**Theorem 1.** *If $p$ is an odd prime, then*

$$v\left(\frac{p-1}{2}\right)^2 \equiv 0, p^3, 2p^3, \text{or } 4p^3 \pmod{11}.$$

$$p_{12}\left(\frac{p-1}{2}\right)^2 \equiv 0, p^5, 2p^5, \text{or } 4p^5 \pmod{19}.$$

**Remark:** If for $1 \leq i \leq 4$ we let $\delta(i)$ denote the Dirichlet density of primes $p$ for which

$$v\left(\frac{p-1}{2}\right)^2 \equiv ip^3 \pmod{11},$$

then

$$\delta(i) = \begin{cases} 3/8 & \text{if } i = 0 \\ 1/3 & \text{if } i = 1 \\ 1/4 & \text{if } i = 2 \\ 1/24 & \text{if } i = 4. \end{cases} \tag{6}$$

The Dirichlet density of primes $p$ for which

$$p_{12}\left(\frac{p-1}{2}\right)^2 \equiv ip^5 \pmod{19}$$

is also given by (6).

To study congruences like (3), Serre and Swinnerton-Dyer used modular Galois representations. If $F(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$ is an eigenform for the action of the Hecke operators and has $\ell$-adic integer coefficients, then by a theorem of Deligne [De], there is a representation $\rho_{\ell,F}$: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}_\ell)$, for each prime $\ell$, depending on $F$ and unramified outside $N\ell$, for which

$$\mathrm{Tr}(\rho_{\ell,F}(\mathrm{Frob}_p)) = a(p)$$

and

$$\det(\rho_{\ell,F}(\mathrm{Frob}_p)) = \chi(p)p^{k-1}$$

for all primes $p \nmid N\ell$. The projectivization of $\rho_{\ell,F}$ reduced modulo $\ell$ is denoted by $\tilde{\rho}_{\ell,F}$.

If the image of $\tilde{\rho}_{\ell,F}$ in $\mathrm{PGL}_2(\mathbb{F}_\ell)$ is $A_4$, $S_4$, or $A_5$, following Swinnerton-Dyer, we say that $\ell$ is *exceptional of type* $(iii)$ for $F$. In particular, when the image is $S_4$, we have

$$a(p)^2 \equiv 0, \chi(p)p^{k-1}, 2\chi(p)p^{k-1}, \text{ or } \ 4\chi(p)p^{k-1} \pmod{\ell} \tag{7}$$

for primes $p \nmid N\ell$. Moreover, the Dirichlet density of primes $p$ for which $a(p)^2 \equiv i\chi(p)p^{k-1} \pmod{\ell}$ is given by (6). If we let $L$ be the kernel field of the projectivization of $\tilde{\rho}_{\ell,F}$, then $a(p)^2 \equiv 0, \chi(p)p^{k-1}, 2\chi(p)p^{k-1}$, or $4\chi(p)p^{k-1} \pmod{\ell}$ precisely when $\mathrm{o}(\mathrm{Frob}_p) = 2, 3, 4$, or $1$ in $\mathrm{Gal}(L/\mathbb{Q}) = S_4$, respectively, where $\mathrm{o}(g)$ denotes the order of $g \in S_4$. Therefore, the density result follows from the Chebotarev Density Theorem.

In the Serre and Swinnerton-Dyer theory of congruences for the coefficients of modular forms on $\mathrm{SL}_2(\mathbb{Z})$, exceptional congruences modulo $\ell$ for $f(z) \in S_k(\Gamma_0(1))$ of types other than type (iii) follow from congruences between two specific modular forms in certain spaces of modular forms of weight no greater than $k+\ell+1$ with coefficients reduced modulo $\ell$. Therefore, these types of exceptional congruences may be simply verified by checking that the $n$th coefficients of the two modular forms in question agree modulo $\ell$ for $n \leq \frac{k+\ell+1}{12}$. It is particularly easy to explicitly calculate many coefficients of the modular forms in these cases. Exceptional congruences of type (iii) are also dictated by congruences between modular forms in spaces of forms with coefficients reduced modulo $\ell$ but are much harder to verify since one of these modular forms is always a weight one form whose explicit construction is difficult. To clarify, observe the following example of a type $(iii)$ congruence that Serre and Swinnerton-Dyer conjectured, but did not prove [SwD, Corollary $(iii)$ to Theorem 4].

**Conjecture.** (*Serre and Swinnerton-Dyer*) *The prime* 59 *is exceptional of type* (*iii*) *for the unique normalized eigenform* $F_{\mathrm{SwD}} \in S_{16}(\Gamma_0(1))$.

In 1983, Haberland gave a constructive proof of the conjecture in a series of three papers [H]. His methods relied heavily on Galois cohomology.

To prove Theorem 1, we exhibit all exceptional congruences of type (*iii*) for eta-product newforms. However, there are a variety of intervening techical questions. Although it is straightforward to eliminate all but finitely many primes $\ell$ as possible exceptional primes, it is harder to verify that any given prime is indeed exceptional. In particular, there could be rogue primes $\ell$ which appear to be exceptional based on limited numerical evidence. More precisely, we do not benefit from the existence of effective forms of the Chebotarev Density Theorem since at the outset we do not have a convenient restriction on the image of the corresponding Galois representation.

Therefore, to prove an exceptional congruence of type (iii) modulo the prime $\ell$ for the coefficients of $f(z) \in M_k(\Gamma_0(N), \chi)$, we first explicitly construct the number field $L$ (e.g., an $S_4$-field if the congruence has form (7)) that seems to be dictating the alleged exceptional congruence. The proof then follows in a purely computational way from recent works of Crespo [Cp1], [Cp2], Quer [Q], and Bayer and Frey [B-F] which show how to explicitly construct odd irreducible complex octahedral two-dimensional Galois representations and the corresponding weight one Hecke newforms coming from the well-known theorems of Weil-Langlands and Langlands-Tunnell [T]. The coefficients of the weight one newform constructed this way satisfy an exceptional congruence of type (iii) modulo $\ell$, so to complete the proof, we employ standard facts about modular forms to compare the well-known properties of the coefficients of the weight one form to the coefficients of $f(z)$. In particular, this strategy may also be applied to prove the Serre and Swinnerton-Dyer example. I. Kiming and H. Verrill [K-V] have also proved the Serre and Swinnerton-Dyer example as well as the exceptional congruences for the coefficients of the modular forms $F_1$ and $F_3$ below using methods somewhat different than ours. It should also be noted that B. Gordon proved exceptional congruences of this type in unpublished notes in the early 1990s. Consider the following eta-product newforms:

$$\begin{aligned}
F_1(z) &= \eta^4(2z)\eta^4(4z) \in S_4(\Gamma_0(8))\\
F_2(z) &= \eta^{-4}(2z)\eta^{16}(4z)\eta^{-4}(8z) \in S_4(\Gamma_0(128))\\
F_3(z) &= \eta^{12}(2z) \in S_6(\Gamma_0(4))\\
F_4(z) &= \eta^{-12}(2z)\eta^{36}(4z)\eta^{-12}(8z) \in S_6(\Gamma_0(64)).
\end{aligned}$$

It is easy to verify that $F_2$ and $F_4$ are $\chi_{-1}$-twists of $F_1$ and $F_3$, respectively.

**Theorem 2.**

(1) *The prime* 11 *is exceptional of type* (*iii*) *for* $F_1$ *and* $F_2$.
(2) *The prime* 19 *is exceptional of type* (*iii*) *for* $F_3$ *and* $F_4$.

**Remark:**

(1) By checking the complete list of eta-product newforms [Ma, Table I], it is straightforward to use (7) to find that the only possible congruences of type (*iii*) are those given in Theorem 2.
(2) There are congruences for the coefficients of eigenforms arising from odd irreducible complex two-dimensional Galois representations for which Artin's Conjecture is true. In particular, we expect many congruences arising from tetrahedral, octahedral, and icosahedral representations. In view of the works of Buhler, Frey, Taylor, and others, one can find newforms of high weight possessing such congruences. However, checking the list of eta-product newforms, and using (7) with $0, 1, 2, 4$ replaced by $0, 1, 4$ and by

$0, 1, 4, \frac{3\pm\sqrt{5}}{2}$ in the tetrahedral and icosahedral cases respectively, we find that there are no congruences for the coefficients of eta-product newforms arising from tetrahedral or icosahedral representations.

Since the only proof of a congruence of type *(iii)* is [H] (to the best of our knowledge), we feel it is important to give a separate and elementary explanation of how any congruence of type *(iii)* may be proved. Moreover, it is interesting to see the Serre and Swinnerton-Dyer theory applied to the eta-product newforms, yielding concrete and elegant number theoretic statements in line with the earlier works cited above.

## 2. Preliminaries.

In this section, we give the important preliminary facts regarding certain kinds of complex two-dimensional Galois representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}),$$

and their connections to weight one newforms. Throughout we will denote by $\rho'$ the projectivization of $\rho$. A representation $\rho$ is *octahedral* if the image of $\rho'$ in $\mathrm{PGL}_2(\mathbb{C})$ is $S_4$. We begin by explicitly constructing $\rho$, an odd irreducible complex octahedral two-dimensional Galois representation.

The projectivization $\rho'$ is uniquely determined by its kernel field which we denote by $K$. The Galois group of $K$ is $S_4$, and $K$ is the splitting field of some irreducible quartic polynomial $g(x)$ having roots $\{x_1, x_2, x_3, x_4\}$. Hence, such a $g(x)$ explicitly defines $\rho'$. Througout this section, we will assume that $K$ contains a quartic subfield with negative discriminant $d$ which we may take to be $K_1 = \mathbb{Q}(x_1)$ without loss of generality. We will also assume that the $A_4$-subfield of $K$ is $M = \mathbb{Q}(\sqrt{-d})$. Later we will discuss how to choose $g(x)$ so that $K$ enjoys certain properties.

Before we can explicitly define $\rho$, it is necessary to determine whether $K$ may be extended to an $\tilde{S}_4$-extension $\tilde{K}$, where $\tilde{S}_4$ is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)$, the double cover of $S_4$ in which transpositions lift to involutions. Assuming the hypotheses on $K$ and $\tilde{K}$ given above, the solvability of this embedding problem may be determined by a simple calculation.

**Theorem 3.** ([B-F, Proposition 1.2]) *The field $K$ may be extended to $\tilde{K}$ if and only if the local invariant $e_{K,p}$ is trivial in the Brauer group of $\mathbb{Q}_p$ for each odd prime $p$ which ramifies in $K$. The invariants $e_{K,p}$ may be calculated from the following table, where $\mathfrak{p}_i, \mathfrak{p}'_i$ and $\mathfrak{p}''_i$ denote factors of residue degree $i$ in $K_1$ and $(d,p)_p = \left(\frac{\alpha}{p}\right)$, where $\alpha = (-1)^{\mathrm{ord}_p(d)} p^{-\mathrm{ord}_p(d)} d$ is the Hilbert Symbol.*

| $pO_{K_1}$ | $e_{K,p}$ |
|---|---|
| $\mathfrak{p}_1^4$ | $(-1)^{\frac{p^2-1}{8}}(-1)^{\frac{p-1}{2}}$ |
| $\mathfrak{p}_1^3\mathfrak{p}'_1$ | $1$ |
| $\mathfrak{p}_1^2\mathfrak{p}'_1\mathfrak{p}''_1$ | $1$ |
| $\mathfrak{p}_1^2\mathfrak{p}'_2$ | $-1$ |
| $\mathfrak{p}_1^2\mathfrak{p}_1'^2$ | $(-1)^{\frac{p-1}{2}}(d,p)_p$ |
| $\mathfrak{p}_2^2$ | $(-1)^{\frac{p+1}{2}}$ |

Assuming that the embedding problem is solvable, an explicit solution $\lambda \in K$ for which $\tilde{K} = K(\sqrt{\lambda})$ is given by a formula of Crespo.

**Theorem 4.** ([Cp1, Theorem 5]) *Let $T$ be the matrix of the bilinear form $\mathrm{Tr}_{K_1(\sqrt{d})/M}(X^2)$ with respect to the basis $\{1, x_1, x_1^2, x_1^3\}$ for $K_1(\sqrt{d})/M$. Let $P$ be a matrix in $GL_4(\mathbb{Q})$ satisfying*

$$P^tTP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2d \end{pmatrix}, \tag{8}$$

*and define*

$$\gamma := \det\left[\begin{pmatrix} 1 & x_1 & x1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{pmatrix} P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2\sqrt{d}} & -\frac{1}{2\sqrt{d}} \end{pmatrix} + I\right] \neq 0, \tag{9}$$

*where $I$ is the identity. Then all solutions of the embedding problem are given by $K(\sqrt{r\gamma})$ as $r$ runs over $\mathbb{Q}^*/\mathbb{Q}^{*2}$.*

The field $\tilde{K}$ is the kernel field of an irreducible complex octahedral two-dimensional Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\tilde{K}/\mathbb{Q}) = \tilde{S}_4 \longrightarrow \mathrm{GL}_2(\mathbb{Z}[\sqrt{-2}]) \subset \mathrm{GL}_2(\mathbb{C}), \tag{10}$$

where we have chosen one of the two faithful irreducible representations of $\tilde{S}_4$ in $\mathrm{GL}_2(\mathbb{C})$. If we suppose that $\rho$ is odd, then by the theorems of Weil-Langlands and Langlands-Tunnell mentioned in the introduction, there exists a weight one Hecke newform $G(z) = \sum_{n=1}^{\infty} b(n)q^n \in S_1(\Gamma_0(N), \chi)$ which is the inverse Mellin transform of the Artin $L$-function $L(s, \rho)$. Here $N$ is the Artin conductor of $\rho$, and $\chi$ is the character $\det(\rho)$. (For background on Artin $L$-functions and connections to weight one newforms, see [Me] or [Se2], for example.) One may compute $N$ directly from its algebraic definition. We verify numerically that $N$ is the conductor of $\rho$ by checking that $G(z)$ is an eigenform for the action of the Atkin-Lehner involution $w(N) = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ with level $N$ as required by the Atkin-Lehner-Li-Miyake theory of newforms.

We note that the image of $\rho$ consists of matrices whose entries are algebraic integers in $\mathbb{Z}[\sqrt{-2}]$. For an arbitrary rational prime $\ell$, the reduction of $\rho$ modulo a prime above $\ell$ in $\mathbb{Z}[\sqrt{-2}]$ has projective image $S_4$, so every prime $\ell$ is exceptional of type (iii) for $G$. It remains to actually calculate the coefficients of $G(z)$. We briefly describe the algorithm of Bayer and Frey enabling ust to explicitly calculate the coefficients $b(p)$ for primes $p \nmid N$.

The field $\tilde{K}$ uniquely determines $\rho$, an irreducible complex octahedral two-dimensional Galois representation, as its kernel field. It will be useful to compute the Artin conductor of $\rho$ and the determinant character $\det(\rho) = \epsilon$. Roughly speaking, the Artin conductor is a product of the ramified primes in $\tilde{K}$, each prime $p$ raised to a certain power depending on the higher ramification groups of $p$ in $\tilde{K}$.

By (10), the coefficients $b(p)$ for primes $p \nmid N$ are captured by the character table for the chosen representation of $\tilde{S}_4$ (see [Df, Chapter 28], for example). The factorization of $p$ in $K_1$ corresponds to a particular conjugacy class of $\mathrm{Frob}_p$ in $S_4$, denoted by $\pi_p$. We denote the conjugacy class of $\mathrm{Frob}_p$ in $\tilde{S}_4$ by $\tilde{\pi}_p$. This information is summarized in the table below, where the lower index on the primes in $K_1$ indicates the residue degree, as before.

Since we know explicitly the factor $\lambda$ for which $\tilde{K} = K(\sqrt{\lambda})$ by Theorem 4, we can determine the conjugacy class in $\tilde{S}_4$ when the factorization types 1, 4, and 5 occur by using Propositions 5 and 6. In what follows, let $\lambda^s$ denote the action of $s \in S_4$ on $\lambda$.

| Factorization type | $pO_{K_1}$ | $\pi_p$ | $\tilde{\pi}_p$ | $b(p)$ |
|---|---|---|---|---|
| 1 | $\mathfrak{p}_1\mathfrak{p}_1'\mathfrak{p}_1''\mathfrak{p}_1'''$ | $1A$ | $1\tilde{A}$ | $2$ |
| | | | $2\tilde{A}$ | $-2$ |
| 2 | $\mathfrak{p}_2\mathfrak{p}_2'$ | $2A$ | $4\tilde{A}$ | $0$ |
| 3 | $\mathfrak{p}_1\mathfrak{p}_1'\mathfrak{p}_2$ | $2B$ | $2\tilde{B}$ | $0$ |
| 4 | $\mathfrak{p}_1\mathfrak{p}_3$ | $3A$ | $3\tilde{A}$ | $-1$ |
| | | | $6\tilde{A}$ | $1$ |
| 5 | $\mathfrak{p}_4$ | $4A$ | $8\tilde{A}$ | $\sqrt{-2}$ |
| | | | $8\tilde{B}$ | $-\sqrt{-2}$ |

**Proposition 5.** ([B-F, Proposition 2.4]) *Suppose $\mathfrak{P}$ is a prime in $K$ above $p$. If $\pi_p = 1A$, (resp. $3A$) then $b(p) = 2$, (resp. $-1$) if and only if $\lambda$ is a square modulo $\mathfrak{P}$.*

**Proposition 6.** ([B-F, Proposition 2.5]) *Suppose $p$ is an odd prime and $pO_{K_1} = \mathfrak{p}_4$. Then $b(p) = \sqrt{-2}$ if and only if*

$$\lambda^{\frac{p-1}{2}} \equiv -\frac{1}{2} + \frac{\lambda^{(3,2,1)} - \lambda^{(1,2,3)}}{2\lambda} \pmod{\mathfrak{p}_4}. \tag{11}$$

## 3. The Proof of Theorem 2.

To prove Theorem 2, we will also need certain facts about modular forms modulo $p$ when $p \geq 5$ is prime. Let $M_{k,p}(\Gamma_0(N))$ denote the $\mathbb{F}_p$-vector space of modular forms in $M_k(\Gamma_0(N)) \bigcap \mathbb{Z}[[q]]$ with coefficients reduced modulo $p$, and suppose that $F(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N)) \bigcap \mathbb{Z}[[q]]$. Then the action of the Ramanujan theta-operator is given by

$$\theta_p(F) :\equiv \sum_{n=0}^{\infty} na(n)q^n \pmod{p} \in M_{k+p+1,p}(\Gamma_0(N)). \tag{12}$$

Propositions 7 and 8 are also useful tools for studying modular forms modulo $p$.

**Proposition 7.**

(1) *If $p$ is an odd prime and*

$$A_p(z) := \frac{\eta^p(z)}{\eta(pz)}.$$

*Then* $A_p(z) \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{(-1)^{\frac{p-1}{2}}p}{\cdot}\right)\right)$*, and*

$$A_p(z) \equiv 1 \pmod{p}.$$

(2) [Sw-D] *Suppose $k \geq 4$ is an even integer. Then the normalized Eisenstein series of weight $k$ with respect to $\mathrm{SL}_2(\mathbb{Z})$ is given by*

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n.$$

*Furthermore, if $p \geq 5$ is prime, then*

$$E_{p-1}(z) \equiv 1 \pmod{p}.$$

*Proof.*

(1) See [G-H], [N2], [N3] for the first part. The second part follows from the Children's Binomial Theorem in characteristic $p$. (i.e., that $1 - x^p \equiv (1-x)^p \pmod{p}$ when $p$ is a prime.)

(2) By the Von Staudt-Claussen Theorem, the power of a prime $p$ dividing the denominator of $B_k$, the $k$th Bernoulli number, is one if $p-1 \mid k$ and zero otherwise. The congruence for $E_{p-1}$ follows.

If $F$ is a number field with ring of algebraic integers $O_F$, if $g = \sum_{n=0}^{\infty} a(n)q^n \in O_F[[q]]$, and if $\mathfrak{p}$ is a prime ideal in $O_F$, then

$$\mathrm{ord}_{\mathfrak{p}}(g) := \begin{cases} \min\{n : \ a(n) \notin \mathfrak{p}\} & \text{if } a(n) \notin \mathfrak{p} \text{ for some } n \\ +\infty & \text{otherwise.} \end{cases}$$

With this definition we can state Sturm's Theorem on the congruence of modular forms.

**Proposition 8.** ([St, Theorem 1]) *If* $f, g \in M_k(\Gamma_0(N)) \bigcap O_F[[q]]$ *and if*

$$\mathrm{ord}_{\mathfrak{p}}(f-g) > \frac{k}{12} \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$

*then* $\mathrm{ord}_{\mathfrak{p}}(f-g) = +\infty$. *i.e.,* $f \equiv g \pmod{\mathfrak{p}}$.

The constant on the right hand side of the inequality in Proposition 8 is called *Sturm's bound* for $f-g$.

To prove that a prime $\ell$ is exceptional of type $(iii)$ for an eigenform $F(z) = \sum_{n=0}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$, we first build a specific odd irreducible complex octahedral two-dimensional Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$ with certain properties depending on $F$, as described in Section 2.

In this general setting, we want to construct $K$ so that it is isomorphic to the kernel field $L$ of $\tilde{\rho}_{\ell,F}$ if $\ell$ were exceptional of type $(iii)$ for $F$. The comments immediately following (7) indicate that we can determine the splitting of primes in $L$ from the coefficients $a(p)$ modulo $\ell$. Any irreducible quartic polynomial defining $K$ must split completely modulo primes which are known to split completely in $L$. The field $K$ must also be unramified outside $N\ell$, which restricts the values of the discriminants of the defining quartics. These requirements allow us to quickly verify whether a given quartic is a good candidate for defining $K$.

Using the facts about modular forms modulo $p$ stated at the beginning of this section, we show that the weight one newform obtained from $\rho$ is congruent modulo $\ell$ to a certain power of the theta-operator acting on $F$, thus proving a nontrivial congruence for the coefficients of $F$. Therefore, $\ell$ is exceptional of type $(iii)$ for $F$.

*Proof of Theorem 2.* We show that 19 is exceptional of type $(iii)$ for $F_3(z) = \eta^{12}(2z)$. We first choose an appropriate irreducible quartic defining the $S_4$-field $K$. Our calculations are based on

$$g(x) = x^4 + 76x^2 - 722x + 1444 \tag{13}$$

with roots $\{x_1, x_2, x_3, x_4\}$ obtained from the arithmetic of the CM elliptic curve 361A [Cr] by a formula of Bayer and Frey [Ba-F, p.401]. The discriminant of $K_1 = \mathbb{Q}(x_1)$ is $-19^3 2^2$, and the $A_4$-subfield is $M = \mathbb{Q}(\sqrt{-19})$. The prime factorization of 19 in $K_1$ is given by $19 = \mathfrak{P}^4$, so $e_{K,19} = 1$, implying that $K$ may be extended to an $\tilde{S}_4$-field $\tilde{K}$ by Theorem 3.

Next, we calculate

$$\begin{aligned}\gamma_{19} =&72x_1^3x_2^2+513x_1^2x_2^2-1368x_1x_2^2-4332x_2^2+1026x_1^3x_2-4104x_1^2x_2+64980x_1x_2\\ &-217683x_2+3420x_1^3+36100x_1^2+488433x_1-2249752\end{aligned} \tag{14}$$

by Theorem 4. The $\tilde{S}_4$-field $K(\sqrt{\gamma_{19}})$ defines an irreducible complex two-dimensional Galois representation $\rho$. Following the Bayer-Frey algorithm, we explicitly construct the coefficients of the weight one Hecke newform $H(z)$ coming from $\rho$. The other solutions $K(\sqrt{r\gamma})$ to this embedding problem define the Galois representations $\rho \otimes \chi_r$, where $\chi_r = \left(\frac{r}{\cdot}\right)$. These representations determine the weight one modular forms $H(z) \otimes \chi_r$. Hence, the weight one form that seems to be dictating the exceptional congruence for the coefficients of $F_3(z)$ is $H(z)$ or some twist of $H(z)$. In particular, the weight one newform we need is

$$G(z) = H(z)\otimes\chi_{-38} = \sum_{n=1}^{\infty} b(n)q^n = q+\sqrt{-2}q^3+q^5+q^7-q^9+\cdots \in S_1\left(\Gamma_0(2^2 19^2), \left(\frac{\cdot}{19}\right)\right) \tag{15}$$

coming from the field $\tilde{K} = K(\sqrt{-38\gamma_{19}})$ having discriminant $2^{18}19^{21}$. The Artin conductor is $2^2 19^2$ and the determinant character is $\left(\frac{\cdot}{19}\right)$, so $\rho$ is also odd.

As mentioned earlier, every prime $p$ is exceptional of type (iii) for $G$. In particular, the prime 19 is exceptional of type $(iii)$ for $G$, so

$$b(p)^2 \equiv 0, p^9, 2p^9, \text{or } 4p^9 \pmod{19}. \tag{16}$$

By Proposition 7,

$$G(z) \equiv G(z)E_{18}(z)^2 A_{19}(z) \pmod{19} \tag{17}$$

and by (12),

$$\theta_{19}^2(F_3(z)) \equiv \sum_{n=0}^{\infty} n^2 a(n) q^n \pmod{19}, \tag{18}$$

where $G(z)E_{18}(z)^2A_{19}(z) \pmod{19}$ and $\theta_{19}^2(F_3(z)) \pmod{19}$ lie in the space $M_{46,19}(\Gamma_0(2^2 19^2))$ whose Sturm bound is 8740. Using Propositions 5 and 6 to calculate the coefficients of $G(z)$, we verify that

$$G(z) \equiv \theta_{19}^2(F_3(z)) \pmod{19} \tag{19}$$

by Proposition 8. In light of (16), this shows that 19 is exceptional of type $(iii)$ for $F_3$, and hence, for $F_4$, since $F_4$ is a $\chi_{-1}$-twist of $F_3$, proving part (2) of Theorem 2.

Table 2 contains data for the remaining case of Theorem 2 (when $\ell = 11$) and for the example of Serre and Swinnerton-Dyer. Here $N$ is the conductor of $\rho$, and $S$ is the Sturm bound for the space $M_{k,p}(\Gamma_0(N))$ in which the relevant congruence of modular forms is proved.

We observe that in each case,

$$F(z) \in M_{\frac{\ell+5}{4}}(\Gamma_0(M)), \tag{20}$$

where $M$ is the part of $N$ prime to $\ell$ and that

$$\theta_\ell^{\frac{\ell-3}{8}}(F(z)) \equiv G(z) \pmod{\ell} \in M_{\frac{\ell^2-1}{8}+1,\ell}(\Gamma_0(N)). \tag{21}$$

A finite computation completes the proof of each exceptional congruence of type (iii).

| $\ell$ | $F(z)$ | $g(x)$ | $N$ | $\chi$ | $S$ |
|---|---|---|---|---|---|
| 11 | $\eta^4(4z)\eta^4(2z)$ | $x^4-16x^3+30x^2+30x-74$ | $2^3 11^2$ | $\left(\frac{\cdot}{11}\right)$ | 2112 |
| 19 | $\eta^{12}(2z)$ | $x^4+76x^2-722x+1444$ | $2^2 19^2$ | $\left(\frac{\cdot}{19}\right)$ | 8740 |
| 59 | $F_{\mathrm{SwD}}$ | $x^4-x^3-7x^2+11x+3$ | $59^2$ | $\left(\frac{\cdot}{59}\right)$ | 2180 |

**Remarks:**

(1) We determine $G(z)$, the appropriate twist of $H(z)$ in the example above, so that $G(z)$ will satisfy (19). In particular, since the $p$th coefficients of such a $G(z)$ (when $p$ is prime) are in the set $\{0, \pm 1, \pm 2, \pm\sqrt{-2}\}$, these $p$th coefficients must be determined by the reduction of the $p$th coefficients of $\theta_{19}^2(F_3(z))$. We use this criteria to check which twist $H(z) \otimes \chi_r$ for squarefree $r \mid 38$ is the correct one.

(2) We note that the choice of $g(x)$ in these examples is certainly not unique. By performing a brute force search on quartic polynomials in the example above, we are immediately led to $g(x) = x^4 - x^3 - 2x^2 - 6x - 2$, having roots $x_1, x_2, x_3, x_4$. In this case, we apply the Bayer-Frey algorithmto $\tilde{K} = K(\sqrt{38\gamma_{19}'})$, where

$$\begin{aligned}\gamma_{19}' =& 6x_1^3x_2^2 + 18x_1^2x_2^2 - 51x_1x_2^2 - 33x_2^2 + 42x_1^3x_2 \\ & - 105x_1^2x_2 + 33x_1x_2 + 3x_2 - 10x_1^3 - 7x_1^2 + 95x_1 + 190\end{aligned}$$

to obtain the same weight one form $G(z)$ as before.

(3) The author notes that the Artin $L$-function constructed in connection with showing that 11 is exceptional of type $(iii)$ for $F_1$ is also constructed by Bayer and Frey [B-F, Example 3], but not in the context of exceptional congruences. In this case, $g(x)$ is obtained from the arithmetic of the CM elliptic curve 121B [Cr].

(4) In Swinnerton-Dyer's original discussion of the exceptional prime 59 for $F_{\mathrm{SwD}}$, the coefficients 11 and 7 in our $g(x)$ are interchanged, a typographical error.

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