HECKE OPERATORS ON CERTAIN SUBSPACES OF INTEGRAL WEIGHT MODULAR FORMS.

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Abstract. Recent works of Garvan [2] and Y. Yang [7], [8] concern a certain family of half-integral weight Hecke-invariant subspaces which arise as multiples of fixed odd powers of the Dedekind eta-function multiplied by $\mathrm{SL}_2(\mathbb{Z})$-forms of fixed weight. In this paper, we study the image of Hecke operators on subspaces which arise as multiples of fixed even powers of eta multiplied by $\mathrm{SL}_2(\mathbb{Z})$-forms of fixed weight.

1. Statement of Results.

We let $\mathfrak{h}$ denote the complex upper half-plane, and we recall the Dedekind eta-function,

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \quad z\in\mathfrak{h}, \quad q := e^{2\pi i z}.$$

In this paper, we study certain subspaces of modular forms constructed using the eta-function. To explain, we first give notation. Let N and k be integers with $N\geq 1$, and let χ be a Dirichlet character modulo N. We let $M_k(\Gamma_0(N),\chi)$ denote the complex vector space of holomorphic modular forms of weight k on $\Gamma_0(N)$ with nebentypus χ, and we let $S_k(\Gamma_0(N),\chi)$ denote the subspace of cusp forms. When $\chi = 1_N$, the trivial character modulo N, we may omit it; when $N=1$, we abbreviate notation with $S_k\subseteq M_k$. We extend notation to half-integral weights in the usual way: When $4\mid N$ and $\lambda\geq 0$ is an integer, we denote by $M_{\lambda+1/2}(\widetilde{\Gamma}_0(N),\chi)$ the space of holomorphic forms of weight $\lambda+1/2$ which transform on $\Gamma_0(N)$ with theta-multiplier and nebentypus χ. For details on modular forms with integral weights, see Section 2 below and the references therein; for details on modular forms with half-integral weights, see [5] and [6]. Furthermore, when $t\in\mathbb{Z}$ is square-free, we let D_t be the discriminant of $\mathbb{Q}(\sqrt{t})$, and we define the Dirichlet character $\chi_t(\cdot) := \left(\frac{D_t}{\cdot}\right)$.

We now precisely describe our setting. We let r be an integer with $1\leq r\leq 23$, and we define $\delta_r := \frac{24}{\gcd(24,r)}$. We note that δ_r is the least positive integer u for which $24\mid ru$. Next, we let ψ_r be the Dirichlet character modulo δ_r defined by

$$\psi_r := \begin{cases}\chi_{-1}^{r/2} & r \text{ even},\\ \chi_3 & \gcd(r,6)=1,\\ 1_{\delta_r} & r\in\{3,9,15,21\}.\end{cases} \tag{1.1}$$

Standard facts on the eta-function imply that

$$\eta(\delta_r z)^r \in \begin{cases} S_{r/2}(\Gamma_0(\delta_r^2),\psi_r), & r \text{ even},\\ S_{r/2}(\widetilde{\Gamma}_0(\delta_r^2),\psi_r), & r \text{ odd}.\end{cases}$$

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For all non-negative even integers s, we define the subspaces

$$S_{r,s} := \{\eta(\delta_r z)^r F(\delta_r z) : F(z) \in M_s\} \subseteq \begin{cases} S_{s+r/2}(\Gamma_0(\delta_r^2), \psi_r), & r \text{ even}, \\ S_{s+r/2}(\widetilde{\Gamma}_0(\delta_r^2), \psi_r), & r \text{ odd}. \end{cases} \tag{1.2}$$

We extend the definition to all even $s \in \mathbb{Z}$ by defining $S_{r,s} = \{0\}$ when $s < 0$, and we note that $M_2 = \{0\}$ implies that $S_{r,2} = \{0\}$.

Recent works of Garvan [2] and Y. Yang [7] prove the Hecke invariance of $S_{r,s}$ when r is odd. These subspaces consist of half-integral weight modular forms. We recall that the Hecke operators with prime index are trivial on half-integral weight spaces. Furthermore, we mention that Yang [8] recently proved, for $\gcd(r, 6) = 1$, that $S_{r,s}$ is Hecke-module isomorphic via the Shimura correspondence to the χ_3-twist of the subspace of newforms in $S_{r+2s-1}(\Gamma_0(6))$ with specified eigenvalues for the Atkin-Lehner involutions W_2 and W_3. Yang proved a similar result for $r \equiv 0 \pmod 3$.

In contrast, r even implies that the subspaces $S_{r,s}$ consist of forms of integer weight. Therefore, for all primes p, the Hecke operators with index p are generally non-trivial. For all even integers $1 \leq r \leq 23$ and all non-negative integers m, our main results describe the image of Hecke operators with index p^m on $S_{r,s}$. For example, we prove, for all even $m \geq 0$, that the Hecke operators with index p^m preserve $S_{r,s}$. The $m = 2$ case is the analogue of the result of Garvan and Yang for even r.

Hecke invariance results of this type can be very useful for explicit computation since $S_{r,s}$ has much smaller $\mathbb{C}$-dimension than the ambient space of cusp forms in which it lies. In particular, we see that $\dim_{\mathbb{C}} S_{r,s} = \dim_{\mathbb{C}} M_s$, which is roughly $s/12$. For odd r, Garvan and Yang used Hecke invariance of $S_{r,s}$ in this way to prove explicit congruences for functions from the theory of partitions such as $p(n)$, the ordinary partition function, and $spt(n)$, Andrews' function which counts the number of smallest parts in all partitions of n.

We require further notation. Let N, k, and χ be as above. For all primes p, we denote the Hecke operator with index p on $M_k(\Gamma_0(N), \chi)$ by $T_{p,k,\chi}$. When the context is clear, we use T_p for $T_{p,k,\chi}$. For all integers $1 \leq v \leq 11$, we define

$$j(v,p) := pv - 12 \left\lfloor \frac{pv}{12} \right\rfloor,$$

the least positive residue of pv modulo 12, and we define

$$t(v,p,k) := k + v - j(v,p).$$

We now state our main theorem.

Theorem 1.1. *Let $1 \leq v \leq 11$ be an integer, let $s \geq 0$ be an even integer, and let p be prime. In the notation above, we have*

$$T_p : S_{2v,s} \longrightarrow S_{2j(v,p),t(v,p,s)}.$$

Example. Let $v = 7$, and let $s \geq 0$ be even. The theorem implies that

$$T_p : S_{14,s} \longrightarrow \begin{cases} S_{14,s}, & p \equiv 1 \bmod 12, \\ S_{22,s-4}, & p \equiv 5 \bmod 12, \\ S_{2,s+6}, & p \equiv 7 \bmod 12, \\ S_{10,s+2}, & p \equiv 11 \bmod 12. \end{cases}$$

Remark. When $p \mid \delta_{2v}$, the operator T_p agrees with Atkin's U_p-operator, and we have $T_p : S_{2v,s} \longrightarrow \{0\}$ since forms in $S_{2v,s}$ have support on exponents coprime to δ_{2v}, as in table (3.3) in Section 3 below.

We also observe that $T_p : S_{2v,s} \longrightarrow \{0\}$ for (v, p, s) as in the table:

v	p	s
$1, 3, 5$	$3 \bmod 4$	$0, 4, 8$
$1, 4, 7$	$2 \bmod 3$	$0, 6$
1	$11 \bmod 12$	12
2	$2 \bmod 3$	$0, 4, 6, 10$

All cases but $(v, p, s) \in \{(5, 3 \bmod 4, 8), (7, 2 \bmod 3, 6)\}$ have $t(v, p, s) < 0$ or $t(v, p, s) = 2$, which give $S_{2j(v,p),t(v,p,s)} = \{0\}$. When $(v, p, s) = (5, 7 \bmod 12, 8)$, we have $t(v, p, s) = 2$, so to verify that $T_p : S_{10,8} \longrightarrow \{0\}$ for primes $p \equiv 3 \pmod 4$, it suffices to consider $(v, p, s) = (5, 11 \bmod 12, 8)$. In this case, the theorem implies that $T_p : S_{10,8} \longrightarrow S_{14,6}$. One may verify that $\eta(12z)^{26} \in S_{2,12}$ has $\eta(12z)^{26} \mid T_5 \neq 0$ in $S_{10,8}$, a one-dimensional space. It follows that $S_{2,12} \mid T_5 = S_{10,8}$. We apply T_p with $p \equiv 11 \pmod{12}$ to obtain

$$S_{10,8} \mid T_p = S_{2,12} \mid T_5 \mid T_p = S_{2,12} \mid T_p \mid T_5 = 0,$$

where the second equality results from commutativity of the Hecke operators and the third results from the third row of the table. A similar argument using $\eta(12z)^{26} \mid T_7 \neq 0$ shows, for primes $p \equiv 2 \pmod 3$, that $T_p : S_{14,6} \longrightarrow \{0\}$.

To conclude the remark, we recall that a q-series $\sum a(n)q^n$ is lacunary if and only if a density one subset of its coefficients vanish. For positive integers r, Serre [4] proved that η^r is lacunary if and only if for all primes $p \equiv 11 \pmod{12}$, we have $\eta^r \mid T_p = 0$. Serre used this result to conclude that η^r is lacunary with $r > 0$ and even if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$. We observe that table entries (v, p, s) above with $s = 0$ imply the lacunarity of η^r with $r \in \{2, 4, 6, 8, 10, 14\}$. The lacunarity of η^{26} follows from the third row of the table.

We now deduce some corollaries.

Corollary 1.2. *In the statement of the theorem, suppose that p is prime with $p \equiv 1 \pmod{\delta_{2v}}$. Then we have $T_p : S_{2v,s} \longrightarrow S_{2v,s}$.*

Proof. The hypothesis on p implies that there exists an integer t with $p = \frac{24t}{\gcd(2v,24)} + 1 = \frac{12t}{\gcd(v,12)} + 1$. It follows that

$$pv = 12t\left(\frac{v}{\gcd(v, 12)}\right) + v \equiv v \pmod{12}.$$

Since $1 \leq v \leq 11$, we see that $v = j(p, v)$ is the least positive residue of pv modulo 12, which proves the statement. ◻

For all $m \geq 0$ and primes p, we define $T_p^m := T_p \circ \cdots \circ T_p$ (m times). With N, k, and χ as above, and with $m \geq 2$, we recall that the Hecke operator with index p^m on $M_k(\Gamma_0(N), \chi)$ is

$$T_{p^m} := T_p \circ T_{p^{m-1}} - \chi(p)p^{k-1}T_{p^{m-2}}. \tag{1.3}$$

The $m = 2$ case of the following corollary is the integer weight analogue of the result of Garvan and Yang.

Corollary 1.3. *Let $1 \leq v \leq 11$, let m and s be non-negative integers with s even, and let p be prime. In the notation above, we have*

$$T_{p^m} : S_{2v,s} \longrightarrow \begin{cases} S_{2v,s}, & m \text{ is even}, \\ S_{2j(v,p),t(v,p,s)}, & m \text{ is odd}. \end{cases}$$

Proof. We proceed by induction on m. When $m = 0$, the map $T_{p^m} = T_0$ is the identity; the case $m = 1$ is Theorem 1.1. We fix $m \geq 2$, and we suppose that the statement of the corollary holds for $0 \leq i \leq m-1$. When m is odd, the induction hypothesis implies that $T_{p^{m-1}}$ fixes $S_{2v,s}$ while T_p and $T_{p^{m-2}}$ map $S_{2v,s}$ to $S_{2j(v,p),t(v,p,s)}$. The statement of the corollary follows from (1.3).

When m is even, the induction hypothesis implies that $T_{p^{m-2}}$ fixes $S_{2v,s}$, and $T_{p^{m-1}}$ maps $S_{2v,s}$ to $S_{2j(v,p),t(v,p,s)}$. Therefore, in view of (1.3), it suffices to show that T_p maps $S_{2j(v,p),t(v,p,s)}$ to $S_{2v,s}$. When $p \mid \delta_{2v}$, the first part of the remark following Theorem 1.1 shows that $T_p : S_{2v,s} \longrightarrow \{0\}$. Hence, for primes $p \nmid \delta_{2v}$, we verify that $j(j(v,p),p) = v$. Since $j(v,p)$ is the least positive residue of pv modulo 12, then $j(j(v,p),p)$ is the least positive residue of $p(pv) = p^2 v \pmod{12}$. We suppose first that $p \geq 5$. We observe that $p^2 \equiv 1 \pmod{12}$ and that $1 \leq v \leq 11$ to conclude that $j(j(v,p),p) = v$. When $2 \nmid \delta_{2v}$, we have $v \in \{4, 8\}$; when $3 \nmid \delta_{2v}$, we have $v \in \{3, 6, 9\}$. In these cases, one also sees that $j(j(v,p),p) = v$. ◻

Next, we let $n \geq 1$ with prime factorization $n = \prod p_i^{e_i}$. We recall that the Hecke operator with index n on $M_k(\Gamma_0(N), \chi)$ is $T_n := \prod T_{p_i^{e_i}}$. The following corollary subsumes Corollary 1.2 and the "even" part of Corollary 1.3.

Corollary 1.4. *Let $1 \leq v \leq 11$, let $s \geq 0$ be an even integer, and let $n \equiv 1 \pmod{\delta_{2v}}$. In the notation above, we have $T_n : S_{2v,s} \longrightarrow S_{2v,s}$.*

The corollary follows from the definition of T_n together with Corollary 1.3. When v has $\delta_{2v} = 12$, we also use the observation that $n \equiv 1 \pmod{12}$ if and only if the multiplicity of all prime factors of n which are congruent to i mod 12 has the same parity for $i \in \{5, 7, 11\}$. For brevity, we omit the details.

Remarks on eigenforms and one-dimensional subspaces. When $s \geq 4$ is an even integer, we recall the Eisenstein series of weight s on $\mathrm{SL}_2(\mathbb{Z})$,

$$E_s(z) := 1 - \frac{2s}{B_s} \sum_{n=1}^{\infty} \sum_{d \mid n} d^{s-1} q^n \in M_s,$$

where B_s is the sth Bernoulli number. We let $E_0(z) := 1$, and we observe that M_s is one-dimensional if and only if $s \in \{0, 4, 6, 8, 10, 14\}$, in which case we have $M_s = \mathbb{C}E_s(z)$. Hence, for such s and for all integers $1 \leq v \leq 11$, we deduce that $S_{2v,s} = \mathbb{C}f_{2v,s}(z)$, with

$$f_{2v,s}(z) := \eta(\delta_{2v} z)^{2v} E_s(\delta_{2v} z). \tag{1.4}$$

For all $n \geq 0$, we define $a_{2v,s}(n) \in \mathbb{Z}$ by $f_{2v,s}(z) = \sum a_{2v,s}(n) q^n$. The remark following Theorem 1.1 together with Corollary 1.2 implies for all primes p, that $f_{2v,s}$ is an eigenform for T_p for all $(2v, s)$ in the set

$$\{(2,0),(4,0),(4,4),(4,6),(4,10),(6,0),(6,4),(6,8),\\ (8,0),(8,6),(12,0),(12,4),(12,6),(12,8),(12,10),(12,14)\}.$$

This fact is well-known; moreover, these forms are normalized newforms. With $b_v := \frac{v}{\gcd(12,v)}$ as in (3.1), Corollary 1.4 together with (2.4) implies, for all $s \in \{0,4,6,8,10,14\}$ and for all $n \equiv 1 \pmod{\delta_{2v}}$, that $f_{2v,s}(z)$ as in (1.4) is an eigenform for T_n with eigenvalue $\lambda_{n,v,s} = \sum \psi_{2v}(d) d^{v+s-1} a_{2v,s}\left(\frac{b_v n}{d^2}\right)$, where the sum is over $d \mid \gcd(b_v, n)$.

The remaining sections of the paper proceed as follows. In Section 2, we provide further necessary facts on modular forms, and in Section 3, we prove Theorem 1.1.

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2. Background on modular forms.

The proof of Theorem 1.1 requires certain facts from the theory of modular forms. For details, see for example [1] and [3].

We first discuss operators on spaces of modular forms. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, let N and k be non-negative integers with $N \geq 1$, let χ be a Dirichlet character modulo N, and let $f(z) = \sum a(n) q^n \in M_k(\Gamma_0(N), \chi)$. We define the slash operator on $f(z)$ by

$$(f \mid_k \gamma)(z) := (\det\gamma)^{k/2}(cz+d)^{-k} f(\gamma z).$$

We define $W_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$; when χ is real, the Fricke involution

$$f \mapsto f \mid_k W_N = N^{-k/2} z^{-k} f\left(\frac{-1}{Nz}\right) \tag{2.1}$$

maps $M_k(\Gamma_0(N), \chi)$ to itself. Next, for all primes p, we define the Hecke operator $T_{p,k,\chi}$. For convenience, we use T_p when the context is clear. These operators map $M_k(\Gamma_0(N), \chi)$ to itself, and they preserve the subspace of cusp forms. On q-expansions, we have

$$\left(\sum a(n) q^n\right) \mid T_{p,k,\chi} = \sum \left(a(pn) + \chi(p) p^{k-1} a\left(\frac{n}{p}\right)\right) q^n, \tag{2.2}$$

with $a\left(\frac{n}{p}\right) = 0$ when $p \nmid n$. When $p \nmid N$ and χ is real, we observe the commutation

$$(f \mid_k W_N) \mid T_p = \chi(p)(f \mid T_p) \mid_k W_N. \tag{2.3}$$

For all positive integers m, we recall the definition of T_n as in the paragraph before Corollary 1.4. One can show that T_n acts on q-expansions in $M_k(\Gamma_0(N), \chi)$ by

$$\left(\sum a(m) q^m\right) \mid T_n = \sum \sum_{d \mid \gcd(m,n)} d^{k-1} \chi(d) a\left(\frac{mn}{d^2}\right) q^m. \tag{2.4}$$

Let $1 \leq v \leq 11$ and let $s \geq 0$ be an even integer. In view of the definition (1.2) of our distinguished subspaces $S_{2v,s}$, we record transformation formulas for $\mathrm{SL}_2(\mathbb{Z})$ on the eta-function and on modular forms in M_s. For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there exists a 24th root of unity, $\epsilon_{a,b,c,d}$, for which

$$\eta\left(\frac{az+b}{cz+d}\right) = \epsilon_{a,b,c,d}(cz+d)^{1/2}\eta(z).$$

We always take the branch of the square root having non-negative real part. As such, one can view the eta-function as a modular form of weight 1/2 on $\mathrm{SL}_2(\mathbb{Z})$ with multiplier system given by $\{\epsilon_{a,b,c,d}\}$. Special cases of this transformation include

$$\eta(z+1) = e^{\frac{2\pi i}{24}}, \quad \eta\left(\frac{-1}{z}\right) = \sqrt{\frac{z}{i}} \cdot \eta(z). \tag{2.5}$$

We also recall, for all $F(z) \in M_s$, that

$$F(z) \mid_s \gamma = F(z). \tag{2.6}$$

3. Proof of Theorem 1.1.

We let $1 \leq v \leq 11$ be an integer. We keep notation from Section 1, and we define

$$b_v := \frac{2v\delta_{2v}}{24} = \frac{v}{\gcd(12, v)}. \tag{3.1}$$

We note that

$$\eta(\delta_{2v}z)^{2v} = q^{\frac{2v\delta_{2v}}{24}} \prod_{n=1}^{\infty}(1-q^{\delta_{2v}n})^{2v} = q^{b_v} + \cdots \in S_v(\Gamma_0(\delta_{2v}^2), \psi_{2v}) \tag{3.2}$$

has order of vanish at infinity equal to b_v, and it has support on exponents congruent to $b_v \pmod{\delta_{2v}}$. The following table gives values of v, b_v, and δ_{2v}.

(3.3)

v	b_v	δ_{2v}
1, 5, 7, 11	v	12
2, 10	$v/2$	6
3, 9	$v/3$	4
4, 8	$v/4$	3
6	1	2

Let $s \geq 0$ be a positive integer, and let $F(z) \in M_s$. Then for all $1 \leq v \leq 11$, we see that $F(\delta_{2v}z)$ has support on exponents divisible by δ_{2v}. Hence, we observe that forms in the subspace $S_{2v,s}$ have support on exponents congruent to $b_v \pmod{\delta_{2v}}$.

Lemma 3.1. *Let v and s be non-negative integers with $1 \leq v \leq 11$ and s even, and let δ_{2v}, b_v, and $S_{2v,s}$ be as above. Suppose that $f(z) \in S_{2v,s}$, that $p \nmid \delta_{2v}$ is prime, and that $f(z) \mid T_p \neq 0$. Then $f(z) \mid T_p$ has support on exponents congruent to $pb_v \pmod{\delta_{2v}}$. Furthermore, $f(z) \mid T_p$ has order of vanish at infinity greater than or equal to the least positive residue of $pb_v \pmod{\delta_{2v}}$.*

Proof. For all integers m, we define $a_f(m)$ by $f(z) = \sum a_f(m)q^m \in S_{2v,s}$. The discussion preceding the lemma shows that $a_f(m) \neq 0$ implies that $m \equiv b_v \pmod{\delta_{2v}}$. Since $p \nmid \delta_{2v}$ and $\delta_{2v} \mid 12$, we have $p \geq 5$, $p = 2$ and $v \in \{4, 8\}$, or $p = 3$ and $v \in \{3, 6, 9\}$. In all cases we find that $p^2 \equiv 1 \pmod{\delta_{2v}}$. Since $f(z) \mid T_p \neq 0$, formula (2.2) shows that there exists $m \geq 0$ with $a_f(pm) \neq 0$ or $p \mid m$ and $a_f\left(\frac{m}{p}\right) \neq 0$ If we suppose that $a_f(pm) \neq 0$, then we have $pm \equiv b_v \pmod{\delta_{2v}}$. Multiplying by p and using $p^2 \equiv 1 \pmod{\delta_{2v}}$ gives $m \equiv pb_v$

$\pmod{\delta_{2v}}$. Similarly, if we suppose that $p \mid m$ and $a_f\left(\frac{m}{p}\right) \neq 0$, then we have $\frac{m}{p} \equiv b_v \pmod{\delta_{2v}}$. Multiplying by p gives $m \equiv pb_v \pmod{\delta_{2v}}$. The second statement follows from the first since $f \mid T_p \neq 0$. □

Proposition 3.2. *Let $1 \leq v \leq 11$ be an integer, and let δ_{2v} and b_v be as above. Suppose that $p \nmid \delta_{2v}$ is prime and that $j \geq 1$ is an integer. We have $\gcd(12, v) \mid j$ and $\frac{2j\delta_{2v}}{24} \equiv pb_v \pmod{\delta_{2v}}$ with $1 \leq \frac{2j\delta_{2v}}{24} \leq \delta_{2v}$ if and only if $j \equiv pv \pmod{12}$ and $1 \leq j \leq 12$. In other words, $\frac{2j\delta_{2v}}{24}$ is the least positive residue of pb_v modulo δ_{2v} if and only if j is the least positive residue of pv modulo 12.*

Proof. First, we suppose that $\gcd(12, v) \mid j$ and $\frac{2j\delta_{2v}}{24} \equiv pb_v \pmod{\delta_{2v}}$. From the definitions, the congruence is equivalent to $\frac{j}{\gcd(12,v)} \equiv p \cdot \frac{v}{\gcd(12,v)} \pmod{\frac{12}{\gcd(12,v)}}$. We multiply by $\gcd(12, v)$ to obtain $j \equiv pv \pmod{12}$.

Conversely, we suppose that $j \equiv pv \pmod{12}$. Then there exists $t \in \mathbb{Z}$ with $12t + pv = j$. It follows that $\gcd(12, v) \mid j$. We divide by $\gcd(12, v)$ in the congruence to obtain $\frac{j}{\gcd(12,v)} \equiv p \cdot \frac{v}{\gcd(12,v)} \equiv pb_v \pmod{\frac{12}{\gcd(12,v)}}$. Since $\frac{j}{\gcd(12,v)} = \frac{2j}{24} \cdot \frac{12}{\gcd(12,v)} = \frac{2j\delta_{2v}}{24}$, we find that $\frac{2j\delta_{2v}}{24} \equiv pb_v \pmod{\delta_{2v}}$. To conclude, we observe that $1 \leq j \leq 12$ holds if and only if

$$1 \leq \frac{2j\delta_{2v}}{24} = \frac{j}{\gcd(12, v)} \leq \frac{12}{\gcd(12, v)} = \delta_{2v}.$$

□

We now prove Theorem 1.1. Let v and s be non-negative integers with $1 \leq v \leq 11$ and s even, and let δ_{2v} and b_v be as above. In view of the remark following the statement of the theorem, we suppose that $p \nmid \delta_{2v}$ is prime. We further recall that $j(v, p)$ is the least positive residue of pv modulo 12 and that $t(v, p, s) = s + v - j(v, p)$. For all $F(z) \in M_s$, we will show that there exists $G_{v,p,F}(z) \in M_{t(v,p,s)}$ such that

$$\frac{(\eta(\delta_{2v}z)^{2v}F(\delta_{2v}z)) \mid T_p}{\eta(\delta_{2v}z)^{2j(v,p)}} = G_{v,p,F}(\delta_{2v}z).$$

For convenience, we define

$$H_{v,p,F}(z) := \frac{(\eta(\delta_{2v}z)^{2v}F(\delta_{2v}z)) \mid T_p}{\eta(\delta_{2v}z)^{2j(v,p)}}.$$

From Lemma 3.1, we see that the function in the numerator has support on exponents congruent to $pb_v \pmod{\delta_{2v}}$. The definition of $j(v, p)$ together with Proposition 3.2 imply that the denominator also has support on exponents in this progression. It follows, for all integers m, that there exists $a_{v,p,F}(m)$ with $H_{v,p,F}(z) = \sum a_{v,p,F}(m)q^{\delta_{2v}m}$. We claim that $G_{v,p,F}(z) := H_{v,p,F}(\frac{z}{\delta_{2v}}) = \sum a_{v,p,F}(m)q^m \in M_{t(v,p,s)}$.

To see this, we first show that $G_{v,p,F}(z)$ transforms with weight $s + v - j(v, p)$ on $\mathrm{SL}_2(\mathbb{Z})$. From its q-series, we see that $G_{v,p,F}(z) = G_{v,p,F}(z + 1)$. It remains to prove that

$$G_{v,p,F}\left(-\frac{1}{z}\right) = H_{v,p,F}\left(-\frac{1}{\delta_{2v}z}\right) = z^{s+v-j(v,p)}H_{v,p,F}\left(\frac{z}{\delta_{2v}}\right) = z^{s+v-j(v,p)}G_{v,p,F}(z).$$

In particular, it suffices to prove that

$$H_{v,p,F}\left(-\frac{1}{\delta_{2v}^2 z}\right) = (\delta_{2v}z)^{s+v-j(v,p)}H_{v,p,F}(z). \tag{3.4}$$

The proof of (3.4) requires some basic facts. First, since $F(z) \in M_s$, we replace z by $\delta_{2v}z$ in (2.6) to obtain

$$F\left(-\frac{1}{\delta_{2v}z}\right) = (\delta_{2v}z)^s F(\delta_{2s}z). \tag{3.5}$$

Next, for all integers $a \geq 1$, we replace z by $\delta_{2v}z$ in (2.5) to get

$$\eta\left(-\frac{1}{\delta_{2v}z}\right)^{2a} = (-i\delta_{2v}z)^a \eta(\delta_{2v}z)^{2a}. \tag{3.6}$$

We now use (2.1), (3.5), and (3.6) to compute

$$\begin{aligned}\left(\eta(\delta_{2v}z)^{2v}F(\delta_{2v}z)\right) |_{v+s}\, W_{\delta_{2v}^2} &= (\delta_{2v}^2)^{-\frac{v+s}{2}} z^{-(v+s)} \eta\left(-\frac{1}{\delta_{2v}z}\right)^{2v} F\left(-\frac{1}{\delta_{2v}z}\right) \\ &= (\delta_{2v}z)^{-(v+s)} \left(-i\delta_{2v}z\right)^v \eta(\delta_{2v}z)^{2v} (\delta_{2v}z)^s F(\delta_{2v}z) \\ &= (-i)^v \eta(\delta_{2v}z)^{2v} F(\delta_{2v}z) = i^{-v}\eta(\delta_{2v}z)^{2v}F(\delta_{2v}z).\end{aligned}$$

For convenience, we let $g_{v,F}(z) := \eta(\delta_{2v}z)^{2v}F(\delta_{2v}z)$. In this notation, the previous calculation reads as

$$g_{v,F}(z) \mid_{v+s} W_{\delta_{2v}^2} = i^{-v} g_{v,F}(z) \tag{3.7}$$

We study the function $(g_{v,F} \mid T_p)(z)$ that occurs in the numerator of $H_{v,p,F}(z)$. Using (2.1), we observe that

$$(g_{v,F} \mid T_p) \mid_{v+s} W_{\delta_{2v}^2} = (\delta_{2v}^2)^{-\frac{v+s}{2}} z^{-(v+s)} (g_{v,F} \mid T_p)\left(-\frac{1}{\delta_{2v}^2 z}\right) = (\delta_{2v}z)^{-(v+s)} (g_{v,F} \mid T_p)\left(-\frac{1}{\delta_{2v}^2 z}\right).$$

It follows that

$$\begin{aligned}(g_{v,F} \mid T_p)\left(-\frac{1}{\delta_{2v}^2 z}\right) &= (\delta_{2v}z)^{v+s}(g_{v,F} \mid T_p) \mid_{v+s} W_{\delta_{2v}^2} = (\delta_{2v}z)^{v+s}\psi_{2v}(p)(g_{v,F} \mid_{v+s} W_{\delta_{2v}^2}) \mid T_p \\ &= \begin{cases} (\delta_{2v}z)^{v+s} i^{-v} g_{v,F}(z) \mid T_p & v \text{ even}, \\ (\delta_{2v}z)^{v+s} i^{p-1-v} g_{v,F}(z) \mid T_p & v \text{ odd}. \end{cases}\end{aligned} \tag{3.8}$$

Since $p \nmid \delta_{2v}$, the commutation (2.3) gives the second equality; we used (1.1) and (3.7) for the third equality, observing that $(-1)^{\frac{p-1}{2}} = \chi_{-1}(p)$.

We turn to the proof of (3.4). We suppose that $1 \leq v \leq 11$ is odd, and we use (3.6) and (3.8) to compute

$$\begin{aligned}H_{v,p,F}\left(-\frac{1}{\delta_{2v}^2 z}\right) &= \frac{(g_{v,F} \mid T_p)\left(-\frac{1}{\delta_{2v}^2 z}\right)}{\eta\left(-\frac{1}{\delta_{2v}z}\right)^{2j(v,p)}} = \frac{i^{p-1-v}(\delta_{2v}z)^{v+s} g_{v,F} \mid T_p}{(-i\delta_{2v}z)^{j(v,p)}\eta(\delta_{2v}z)^{2j(v,p)}} \\ &= \frac{i^{p-1-v}(\delta_{2v}z)^{v+s-j(v,p)} g_{v,F} \mid T_p}{i^{-j(v,p)}\eta(\delta_{2v}z)^{2j(v,p)}} = i^{p-1-v+j(v,p)}(\delta_{2v}z)^{v+s-j(v,p)} H_{v,p,F}(z).\end{aligned}$$

Since v is odd and $p \nmid \delta_{2v}$, we see that p is odd. We recall that $j(v,p) \equiv pv \pmod{12}$ to find that

$$p - 1 - v + j(v,p) \equiv p - 1 - v + pv \equiv (v+1)(p-1) \equiv 0 \pmod 4.$$

Similarly, when $1 \leq v \leq 11$ is even, we compute

$$H_{v,p,F}\left(-\frac{1}{\delta_{2v}^2 z}\right) = i^{-v+j(v,p)}(\delta_{2v}z)^{v+s-j(v,p)}H_{v,p,F}(z).$$

In this case, we have

$$-v + j(v,p) \equiv -v + pv \equiv v(p-1) \pmod 4.$$

When p is odd, we have $p-1$ is even; then v and $p-1$ even imply that $-v+j(v,p) \equiv 0 \pmod 4$. When $p = 2$, we have $v \in \{4, 8\}$ implies that $v + j(v,p) \equiv 0 \pmod 4$. Hence, we conclude (3.4) in all cases. We replace z by $\frac{z}{\delta_{2v}}$ in (3.4) to see that $G_{v,p,F}(-1/z) = z^{v+s-j(v,p)}G_{v,p,F}(z)$. Since $z \mapsto z+1$ and $z \mapsto -\frac{1}{z}$ generate $\mathrm{SL}_2(\mathbb{Z})$, it follows that $G_{v,p,F}(z)$ transforms with weight $t(v,p,s) = v + s - j(v,p)$ on $\mathrm{SL}_2(\mathbb{Z})$.

To finish the proof of the lemma, we show that $G_{v,p,F}(z)$ is holomorphic on $\mathfrak{h}$ and at the cusp infinity. Since $G_{v,p,F}(z) = H_{v,p,F}\left(\frac{z}{\delta_{2v}}\right)$, it suffices to prove the same statement with $G_{v,p,F}(z)$ replaced by $H_{v,p,F}(z)$. Since $(\eta(\delta_{2v}z)^{2v}F(\delta_{2v}z)) \mid T_p$ is holomorphic on $\mathfrak{h}$ and $\eta(z) \neq 0$ on $\mathfrak{h}$, we conclude that $H_{v,p,F}(z)$ is holomorphic on $\mathfrak{h}$. When $(\eta(\delta_{2v}z)^{2v}F(\delta_{2v}z)) \mid T_p \neq 0$, Lemma 3.1 implies that it has order of vanish at infinity greater than or equal to the least positive residue of $pb_v \pmod{\delta_{2v}}$. By Proposition 3.2 and the definition of $j(v,p)$, this least residue has value $\frac{2j(v,p)\delta_{2v}}{24}$. Since $\eta(\delta_{2v}z)^{2j(v,p)}$ has order of vanish at infinity equal to $\frac{2j(v,p)\delta_{2v}}{24}$, we deduce that $H_{v,p,F}(z)$ is holomorphic at infinity.

References

[1] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, 228, Springer, New York, 2005. MR2112196 (2006f:11045)

[2] F. G. Garvan, Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank, Int. J. Number Theory **6** (2010), no. 2, 281–309. MR2646759 (2011j:05032)

[3] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, 17, Amer. Math. Soc., Providence, RI, 1997. MR1474964 (98e:11051)

[4] J.-P. Serre, Sur la lacunarité des puissances de η, Glasgow Math. J. **27** (1985), 203 - 221.

[5] G. Shimura, On modular forms of half integral weight, Ann. of Math. (2) **97** (1973), 440–481. MR0332663 (48 #10989)

[6] G. Shimura, *Modular forms: basics and beyond*, Springer Monographs in Mathematics, Springer, New York, 2012. MR3025356

[7] Y. Yang, Congruences of the partition function, Int. Math. Res. Not. IMRN **2011**, no. 14, 3261–3288. MR2817679 (2012e:11177)

[8] Y. Yang, Modular forms for half-integral weights on $\mathrm{SL}_2(\mathbb{Z})$, Nagoya Math. J., to appear.

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